New Redundant Representations of Complex Numbers and Vectors

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Abstract

In this paper, we present a new redundant representation for complex numbers, called ***polygonal representation****. This representation enables fast carry-free addition (in a way quite similar to the carry-free addition in signed-digits number systems), and is convenient for multiplication. Then we generalize our technique in order to handle n-dimensional vectors.*

Introduction

Complex numbers and vectors are used in various fields of Computer Science. We need to represent these objects as efficiently as possible. The most common way is of course to represent them as arrays of real numbers: here we shall study an other way, using complex digit sets (in the real case, digit sets have been widely studied, for instance by Matula [1], Petkovsek [2], Carter and Robertson [3]).

Our goal is to generalize the carry-free addition technique of Avizienis [4] for signed-digit arithmetic to complex and vectorial arithmetic. As in Avizienis' number systems, we need *redundancy* in order to enable carry free addition. In the first part of this paper, we deal with manipulation of Complex Numbers. Then we shall generalize our technique to the manipulation of vectors.

1 Polygonal representation of complex numbers

1.1 Usual representation

The most common way is of course to represent the complex number $a + i.b$ by the couple (a, b) of real numbers. This representation has some drawbacks: the sets of numbers of the form $a+ib$ with a and b in real intervals (rectangles) are not stable by complex multiplication: for example $E = \left\{a + i.b/(a, b) \in [-1, 1]^2\right\}$ is represented in *Fig. 1* with the set $F = \left\{xy/(x, y) \in E^2\right\}$

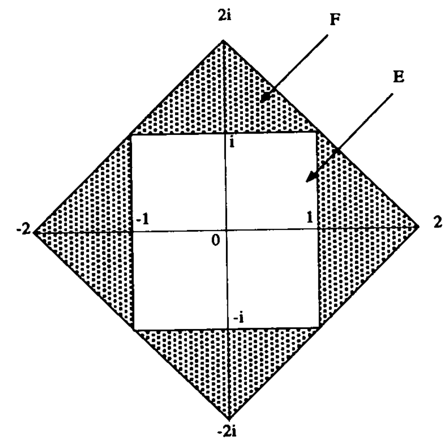


Fig. 1: Sets E and F

Some other representations have been previously proposed:

- Radix $i\sqrt{2}$ with digits in $\{0, 1\}$
- Radix $i - 1$ with digits in $\{0, 1\}$ [5].

These two representations use as radix a 8^{th} root of 16 (of modulus $\sqrt{2}$). Both representations use a complex radix and integer digits: in a dual way, we shall use here an integer radix and complex digits.

1.2 Use of p^{th} roots of unity and zero as digits

Assume that we are in radix 2. We shall consider the case where the digits are chosen in the set containing the p^{th} roots of the unity and zero:

$$D_p = \left\{0, 1, \omega, \omega^2, \cdots, \omega^{p-1}\right\} \text{ where } \omega = e^{\frac{2i\pi}{p}}$$

As in conventional number systems, a number x is represented by a digit sequence $(d_i), d_i \in D_p$ which satisfies $x = \sum_{i=-\infty}^{\infty} d_i 2^i$.

i. p=1: It is the usual representation of real numbers in radix 2 with digits in $\{0, 1\}$. Each positive real number is representable.

ii. p=2: We obtain the binary signed-digit representation of real numbers, with digits in $\{-1, 0, 1\}$ [4]. Each real number is representable.

iii. p=3: The digits are taken in $\{0, 1, j, j^2\}$, with $j = e^{2i\pi/3}$. The set of representable numbers has a fractal structure. *Fig 2* presents the set of numbers representable only with "fractional" digits, i.e. the set of numbers of the form $x = \sum_{i=0}^{\infty} d_i 2^{-i}$.

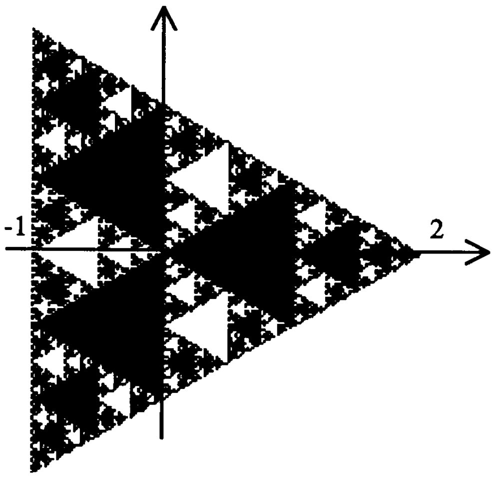


Fig. 2: Numbers representable in radix 2 with digits in $\{0, 1, j, j^2\}$

iv. p≥4: If $p \geq 4$, every complex number can be represented in radix 2 with digits in D_p. As a proof, we give for $p = 4$ an algorithm which computes a representation of a given number. This algorithm may be easily extended to higher values of p (see [6]).

Assume that we want to compute a representation of a number x. Since from a representation of x, a representation of $2^k x$ may be obviously deduced, we assume without loss of generality, that x lies into the square S delimited by numbers $2, 2i, -2, -2i$ (see *Fig. 3*). We divide this square into the 5 areas labelled $1, -1, i, -i$ and 0 depicted in *Fig. 3*.
Let us denote $x^{(0)} = x$. The sequence d_i of digits of a representation of x is defined by induction as:

- $d_i = k$ if $x^{(i)}$ lies in the area labelled k
- $x^{(i+1)} = 2(x^{(i)} - k)$

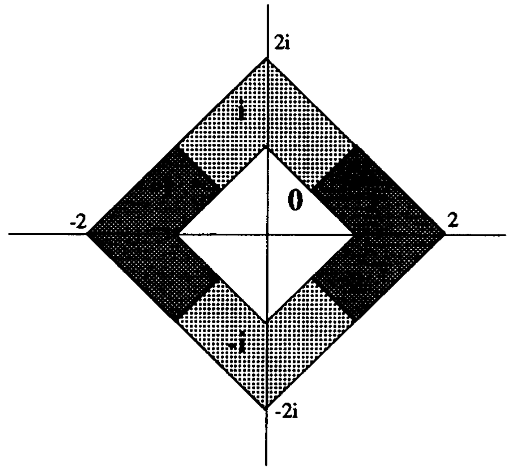


Fig. 3: The square S

1.3 Hexagonal binary representation

i. Definitions: Now let us consider the representation of complex numbers in radix 2 with digits in D_6. This representation will give us a compact encoding of complex numbers. Moreover, it offers the ability of performing fully parallel addition in constant time (i.e. independent from the length of the operands). Let us denote $H(1) = D_6$. By extension, $H(n)$ is the set of the points which are sum of n elements of $H(1)$ (examples are shown *Fig. 4)*. Then $H(m) + H(n) = \{x + y / x \in H(m), y \in H(n)\}$ satisfies $H(m) + H(n) = H(m + n)$, and $H(m) * H(n) = \{xy / x \in H(m), y \in H(n)\} \subset H(m * n)$. Finally, we define $H(\infty) = H$.

Let us denote $h(a) = \{0, 1, 2, \cdots, a\}$. Since $D_6 = H(1) = \{0, 1\} + j.\{0, 1\} + j^2\{0, 1\}$, we deduce that $H(a) = h(a) + j.h(a) + j^2.h(a)$. Therefore H may be written as $\mathbb{N} + j\mathbb{N} + j^2\mathbb{N}$. Therefore any element of H may be represented in radix 2 with digits in $H(1)$. (Notice that H is a lattice of the complex field $\mathbb{C}$.)

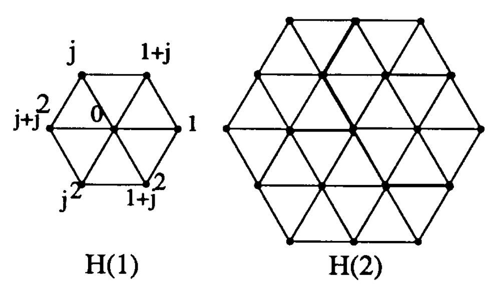


Fig. 4: Sets H(1) and H(2).

ii. Representation of the elements of H: In a similar fashion, we can represent any element x of H in radix b with digits in $H(b-1)$:

$$x = \sum_{i=0}^{\infty} d_i b^i, \quad d_i \in H(b-1)$$

In the following, we deal with representations of H in radix b with digits in $H(a)$, $a \leq b-1$. We shall study what conditions must be satisfied by a and b in order to represent H and to perform fully parallel additions without carry propagation. Before doing that, let us propose a convenient way to represent the "hexagonal digits" of $H(a)$: an "hexagonal digit" d of $H(a)$ is represented by 3 elements d^1, d^2 and d^3 of $h(a) = \{0, 1, 2, \cdots, a\}$, satisfying $d = d^1 + d^2 j + d^3 j^2$. This representation has some advantages, including:

- If $d = d^1 + d^2 j + d^3 j^2$ then $-d = (a - d^1) + (a - d^2) j + (a - d^3) j^2$
- If $d = d^1 + d^2 j + d^3 j^2$ then $\overline{d} = d^1 + d^3 j + d^2 j^2$ ($\bar{d}$ is the complex conjugate of d)

Theorem 1.
H can be represented in radix b with digits in $H(a)$, $a \leq b-1$, if $3(a+1) > 2b$. ($a \in \mathbb{N}$, $b \in \mathbb{N}$)

Proof:
We just need to prove that each element of $H(b)$ can be represented, since a representation of a number $x.b^k$ is deducible immediately from a representation of x. It suffices to show that $H(b)$ is covered by the seven sets $H(a)$, $H(a)+b$, $H(a)+\omega b$, $H(a)+\omega^2 b$, $H(a)+\omega^3 b$, $H(a)+\omega^4 b$ and $H(a)+\omega^5 b$ (see *Fig. 5*).

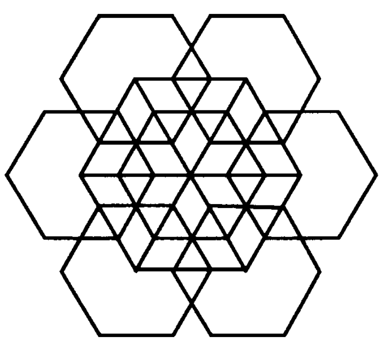


Fig. 5: Covering of H(b)

Therefore, from geometrical reasons, it suffices that $a \geq 2b/3$, as depicted *Fig. 6*. Since the sets $H(b)$ and $H(a)$, $H(a)+b$, $H(a)+\omega b$, ... are *discrete* sets, the condition $a \geq 2b/3$ is equivalent to the condition $a > 2b/3 - 1$.

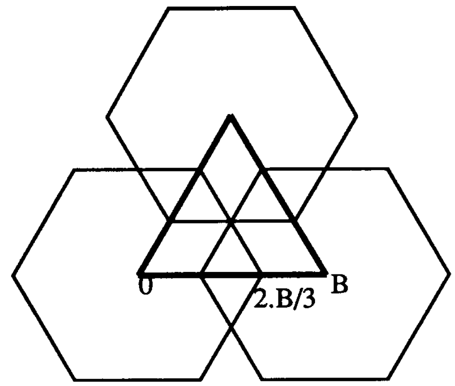


Fig. 6: Majoration of a

iii. Fully parallel addition in radix b with digits in H(a): We give an algorithm derived from the technique of Avizienis [4], in order to add elements of H written in radix b with digits in $H(a)$.

Let X and Y be in H, $X = \sum_{i=0}^{N} x_i b^i$ and $Y = \sum_{i=0}^{N} y_i b^i$. For each integer i, $x_i + y_i$ belongs to $H(2a)$: we find $c_{i+1} \in H(2)$ and $s_i \in H(a-2)$ such that $x_i + y_i = b.c_{i+1} + s_i$. The value $t_i = c_i + s_i$ belongs to $H(a)$, and $X + Y$ is obviously equal to $\sum t_i b^i$, therefore, if we are able to compute the values c_{i+1} and s_i (i.e. if $H(2a) \subset bH(2) + H(a-2)$), then we are able to perform an addition.

Theorem 2.
If $3((a-2)+1) > 2b$, then $H(2a) \subset bH(2) + H(a-2)$, therefore a fully parallel addition is possible in radix 2 with digits in $H(a)$.

Proof:
Since $a \leq b$ then $H(2a) \subset H(2b)$. From geometrical considerations, $H(2b)$ is equal to $bH(1) + H(b)$ (see *Fig. 7*).

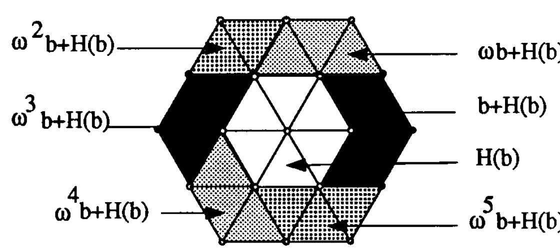


Fig. 7: H(2b)=bH(1)+H(b)

If $3((a-2)+1) > 2b$ then from theorem 1, $H(b) \subset bH(1) + H(a-2)$. Therefore $H(2a) \subseteq bH(2) + H(a-2)$. From $a \leq b-1$ and $3((a-2)+1) > 2b$, we deduce that if $b > 6$ there exists a fully parallel addition algorithm.

1.4 Hadwired fully parallel adder in H

We present in *Fig. 8* an hadwired fully parallel adder for hexagonal binary representation, divided in slices. We can notice that in fact, this adder works in radix 8. We perform the addition $Z = X + Y$ with $x_i = a_i + b_i j + c_i j^2, y_i = a_i^{'} + b_i^{'} j + c_i^{'} j^2, z_i = A_i + B_i j + C_i j^2$ In *Fig. 8*, the terms a_i, b_i, c_i are permuted in order to obtain 3 identical slices.

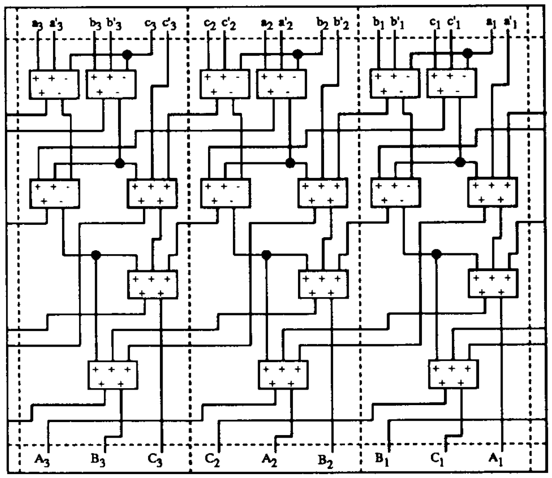


Fig. 8: A 3–digit (or 1–radix 8 digit) slice of the adder.

This adder uses the elementary cells described below. One of these cells is well known (it is a *Full Adder*), and the other cell is quite similar.

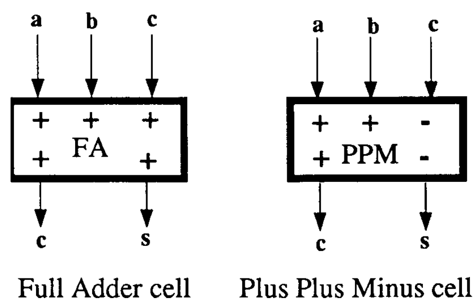


Fig. 9: The elementary cells of the adder. Those cells compute $a + b \pm c = 2c \pm s$

2 Addition of vectors

In this part, we define techniques for adding n-dimentional vectors in time independent from the number of digits used to represent each component of the vectors. We define redundant representation of vectors and carry free addition algorithm.

2.1 Definitions

We define a redundant representation of an n-dimentional vector as a decomposition of this vector in a system $e = (e_1, e_2, ..., e_{n+1})$ satisfying two conditions:

1. $e_1 + e_2 + ... + e_{n+1} = 0$
2. n vectors of the system e are always a basis.

Each vector of R^n is defined by $n+1$ non unique positive real numbers $x^1, ..., x^{n+1}$ such as $x = \sum_{i=1}^{n+1} x^i e_i$. If $x^i = \sum_{j=-\infty}^{+\infty} x_j^i b^j$, then $x = \sum_{j=-\infty}^{+\infty} X_j b^j$, where $X_j = \sum_{i=1}^{n+1} x_j^i e_i$ is called a **"vectorial digit"**.

We denote H the subset of vectors of R^n whose coordinates are integers:

$$H = \left\{ x = \sum_{i=1}^{n+1} x^i e_i / \forall i \; x^i \in N \right\}$$

These notions generalize those presented in part A, where $n = 2$ and $e = (1, j, j^2)$.

The norm of a given $x = \sum_{i=1}^{n+1} x^i e_i$, where x^i is positive or negative, is defined as

$$|x| = \max_i x^i - \min_i x^i$$

We define a distance d as $d(x,y) = |x - y|$.

If each real is written in radix b with digits in $\mathrm{h}(\omega) = \{0, 1, ..., \omega\}$, then each vector is written in radix b with vectorial digits in $\mathbf{H}(\omega) = \{\mathbf{x} \in \mathbf{H} / |\mathbf{x}| \le \omega\}$. The set $H(a)$ satisfies the property $H(a) + H(b) = H(a + b)$. $'a'$ is called radius of $H(a)$

Our addition algorithm is a generalization of Avizienis' algorithm [4]: the sum of two vectorial digits X_i and Y_i in $H(\omega)$ must be decomposed as a sum $bC_{i+1} + S_i$, where C_{i+1} in $H(c)$ is a carry and S_i in $H(s)$ is a partial sum, and $c + s = \omega$. We have to solve two problems: first, we need to find c and s such that $H(2\omega) \subset bH(c) + H(s)$ and $c + s = \omega$; second, we need to find an algorithm to decompose each vectorial digit of $H(2\omega)$ in $bH(c)$ and $H(s)$.

If we call **"interior radius"** of a set X the highest integer a such that X contains $H(a)$,

$$r_{int}(X) = max\{a / H(a) \subset X\}$$

then $H(2\omega) \subset bH(c) + H(s)$ iff
$2\omega \leq r_{int}(bH(c) + H(s)) = r_{int}$.

We call **"recover radius"** of a set $X = \{x^1, ..., x^q\}$ the value:

$$r_{rec}(X) = min\{a/[X] \subset X + H(a)\} = \max_{x\in[X]} d(x, X)$$

where $[X]$ is defined by

$[X] = \{x \in H/\exists\lambda_1, ..., \lambda_q \geq 0, \sum_{i=1}^{q} \lambda_i = 1$ and $x = \sum_{i=1}^{q} \lambda_i x^i\}$.

2.2 Preliminary results

We have the following results:

Theorem 3.

(1) $r_{rec} = r_{rec}(bH(c)) = \left\lfloor \frac{nb}{n+1} \right\rfloor$
(2) $r_{int} = s$ if $s < r_{rec}$
(3) $r_{int} \geq bc$ if $s \geq r_{rec}$
(4) if $n \geq 2$ then
$H(2\omega) \subset bH(c) + H(s) \;\Rightarrow\; c \geq 2$
(5) if $n = 1$ then
$s \geq r_{rec} \;\Rightarrow\; r_{int} = bc + s$

Proof

We have an immediate result: $[bH(c)] = H(bc)$ and $r_{int} \geq s$

(1) is proved if we show:

$i \quad \forall z \in [bH(c)]\; d(z, bH(c)) \leq \left\lfloor \frac{nb}{n+1} \right\rfloor$
$ii \quad \exists z \in [bH(c)],\; d(z, bH(c)) = \left\lfloor \frac{nb}{n+1} \right\rfloor$

To this purpose, we first show the following lemma:

Lemma

Let $z \in [bH(c)]$

$\exists P = \{P_0, ..., P_n\} \subset bH(c)$ such that

$$z \in [P]$$
$$d(z, bH(c)) = d(z, P)$$
$$\sum_{i=0}^{n} d(z, P_i) = nb$$

Let $z = \sum_{i=1}^{n+1} z^i e_i$ with $z^i \geq 0$

and $z^i = b\alpha_i + \beta_i$ where $0 \leq \alpha_i \leq c - 1, \;\; 0 \leq \beta_i \leq b$

then, reordering the β_i such that $b \geq \beta_{i_1} \geq \beta_{i_2} \geq ... \geq \beta_{i_{n+1}}$, we obtain

$$z = P_0 + \sum_{j=1}^{n+1} \beta_{i_j} e_{i_j} \qquad \text{where } P_0 = \sum_{i=1}^{n+1} b\alpha_i e_i$$
$$= P_0 + \sum_{j=1}^{n} (\beta_{i_j} - \beta_{i_{n+1}}) e_{i_j} \text{ since } \sum_{j=1}^{n+1} e_{i_j} = 0.$$

Let $\beta_{i_n} - \beta_{i_{n+1}} = b\lambda_n$,
$\beta_{i_j} - \beta_{i_{n+1}} = (\beta_{i_{j+1}} - \beta_{i_{n+1}}) + b\lambda_j$

$\beta_{i_j} - \beta_{i_{j+1}} \geq 0 \;\Rightarrow\; \lambda_j \geq 0$

$\beta_{i_1} - \beta_{i_{n+1}} \leq b \;\Rightarrow\; \lambda_1 + \lambda_2 + ... + \lambda_n \leq 1$

Let $\lambda_0 = 1 - (\lambda_1 + \lambda_2 + ... + \lambda_n)$

then

$$z = \sum_{j=0}^{n} \lambda_j \left(P_0 + b\sum_{k=1}^{j} e_{i_k} \right) = \sum_{j=0}^{n} \lambda_j P_j$$
$$\text{where } P_j = P_0 + b\sum_{k=1}^{j} e_{i_k} = P_{j-1} + be_{i_j}.$$

By induction we can show that:
$z - P_j = b \sum_{k=1, j\geq 2}^{j-1} \left(\sum_{l=k}^{j-1} \lambda_l \right) e_{i_k} + b \sum_{k=j+1}^{n+1} \left(\sum_{l=0}^{j-1} \lambda_l + \sum_{l=k, k\leq n}^{n} \lambda_l \right) e_{i_k}$.

If we set $d_i = d(z, P_i)$ then, using the previous formula, $d_i = b\sum_{j\neq i} \lambda_j = b(1 - \lambda_i)$ and $\sum_{i=0}^{n} \lambda_i = 1 \Rightarrow \sum_{i=0}^{n} d_i = nb$. The point $z = \sum_{i=0}^{n} \lambda_i P_i = \sum_{i=0}^{n} \left(1 - \frac{d_i}{b}\right) P_i$ is completely defined by the set P and a choice of $d_0, ..., d_n$ such that $\sum_{i=0}^{n} d_i = nb$

An other point Q in $bH(c)$ can be written:
$Q = P_0 + \sum_{j=1}^{n} \gamma_j (P_j - P_0)$ where the γ_j are all integers.
We always have $d(z, Q) \geq min(d_i) = d(z, P)$, and the result $d(z, bH(c)) = d(z, P)$ holds.

We denote $P(P_0, b; e_{i_1}, e_{i_2}, ..., e_{i_n}) = \{P_0, ..., P_n\}$

Let $z \in H(bc)$. Applying the lemma
$\exists P = \{P_0, ..., P_n\} \subset bH(c),\;\; d(z, bH(c)) = d(z, P)$.

Let $d_i = d(z, P_i)$

$d(z, bH(c)) = d(z, P) = min(d_i) \leq \left\lfloor \frac{nb}{n+1} \right\rfloor$ and the point z defined by

$$d_0 = \ldots = d_{q-1} = \left\lceil \frac{nb}{n+1} \right\rceil$$
$$d_q = \ldots = d_n = \left\lfloor \frac{nb}{n+1} \right\rfloor \text{ where } nb \equiv q \mod(n+1)$$

satisfies $d(z, bH(c)) = min(d_i) = \left\lfloor \frac{nb}{n+1} \right\rfloor$. So (1) is proved.

To prove (2) we show that if $s \leq r_{rec} - 1$ then $\exists z \in H(s+1),\ d(z, bH(c)) > s$.

$z \in [P(0, b; e_1, \ldots, e_n)]$ defined by

$$d_0 = s + 1$$
$$d_1 = \ldots = d_q = \left\lceil \frac{nb - (s+1)}{n} \right\rceil$$
$$d_{q+1} = \ldots = d_n = \left\lfloor \frac{nb - (s+1)}{n} \right\rfloor$$
$$\text{where } nb - (s+1) \equiv q \mod(n)$$

is such that $d(z, bH(c)) > s$ since $d(z, bH(c)) = \min_i (d_i)$.

(3) is a consequence of the result $[bH(c)] = H(bc)$

To prove (4) we show that $H(2\omega) \not\subset bH(1) + H(s)$. In order to do that, let us consider the point $z = (b+s)e_1 + (s+1)e_2$, which belongs obviously to $H(2\omega)$. Let us show that for any $y \in bH(1)$, $d(z, y) > s$. y may be written $b \sum_{i=1}^{n+1} \epsilon_i e_i$ with $\epsilon_i = 0, 1$.

If $\epsilon_2 = 1$ then $d(y, z) \geq 2b - b\epsilon_1 - 1 \geq b - s > s$

If $\epsilon_2 = 0$ then $d(y, z) \geq b\epsilon_3 + s + 1 \geq s + 1 > s$

(5) is obvious.

2.3 Determination of b, ω and s

Now we can determine b, ω, s such that $H(2\omega) \subset bH(c) + H(s)$.

The relation $H(2\omega) \subset bH(c) + H(s)$ is equivalent to $2\omega \leq r_{int}$. Since $2\omega = 2(b-1) > s$, r_{int} may be greater than s, so s may be greater than r_{rec}:

$$H(2\omega) \subset bH(c) + H(s) \Leftrightarrow \begin{cases} s \geq r_{rec} \\ 2\omega \leq r_{int} \end{cases}$$

If $n \geq 2$, $2\omega = 2b - 2 \leq 2b \leq bc$: the relation $2\omega \leq r_{int}$ always holds. The relation $s \geq r_{rec}$ gives $(s+1)(n+1) > nb$. With $c + s = \omega = b - 1$ and $c \geq 2$, we obtain $s + 1 \leq b - 2$ and then $b > 2n + 2$. This result shows again that in the complex field the basis should be greater than 6 (theorem 2).

We can choose

$$b > 2n + 2$$
$$s = \left\lfloor \frac{nb}{n+1} \right\rfloor \quad c = b - s$$

If $n = 1$, we have a second relation $2\omega \leq r_{int} = bc + s$ and we deduce $b \geq 3$.

2.4 Circuit building method

Now let us deal with a circuit able to compute the sum of two vectors. We generalize the circuit used to add complex numbers presented in figure 8.

We want to add two n-dimensional vectors x and y, written in radix $b = 2^{n+1}$ with p vectorial digits in $H(b-1)$:

$$z = x + y = \sum_{j=0}^{p(n+1)-1} \left(\sum_{i=1}^{n+1} (x_j^i + y_j^i) e_i \right) 2^j$$

We use a matrix Z whose term Z_{ij} counts the number of terms in $e_i 2^j$ we need to add. So $Z_{ij} = 2\ \forall i, j$.

For instance:

$n = 2 \quad p = 3 \quad b = 8$

$$Z = \begin{array}{c} \begin{array}{ccc} \quad b^2 \quad & \quad b^1 \quad & \quad b^0 \quad \end{array} \\ \begin{pmatrix} \overbrace{2\ \ 2\ \ 2} & \overbrace{2\ \ 2\ \ 2} & \overbrace{2\ \ 2\ \ 2} \\ 2\ \ 2\ \ 2 & 2\ \ 2\ \ 2 & 2\ \ 2\ \ 2 \\ 2\ \ 2\ \ 2 & 2\ \ 2\ \ 2 & 2\ \ 2\ \ 2 \end{pmatrix} \end{array} \begin{array}{c} \\ e_3 \\ e_2 \\ e_1 \end{array}$$

We split this matrix into p blocks: The i^{th} blocks corresponds to the terms in b^i. Each block is equal to a $(n+1)(n+1)$ matrix Z^0

$$Z^0 = \begin{array}{c} \begin{array}{ccc} 2^2 & 2^1 & 2^0 \end{array} \\ \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \end{array} \begin{array}{c} \\ e_3 \\ e_2 \\ e_1 \end{array}$$

We can apply the following transformations:

1. Vertical transformation:
   In a column, one term is redistributed from its place to the other places of the same column. This transformation is based upon the relation $\sum_{i=1}^{n+1} e_i = 0$
2. Horizontal transformation:
   A term equal to 3, may be transformed in a term equal to 1 in the same place, and a carry transferred to the same place in next column.

For instance the vertical transformation transforms Z^0 into $W^0 = \begin{pmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix}$, while the horizontal transformation transforms W^0 into

$$Z^1 = \begin{matrix} 1 \leftarrow \\ 1 \leftarrow \\ 1 \leftarrow \end{matrix} \begin{pmatrix} 1+1 \leftarrow & 1+1 \leftarrow & 1 \\ 1 & 1+1 \leftarrow & 1+1 \\ 1+1 \leftarrow & 1 & 1+1 \end{pmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \end{matrix} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 2 \end{pmatrix}$$

These transformations are implemented using PPM or FA Cell (see figure 9). It leads to the circuit *Fig. 10.*

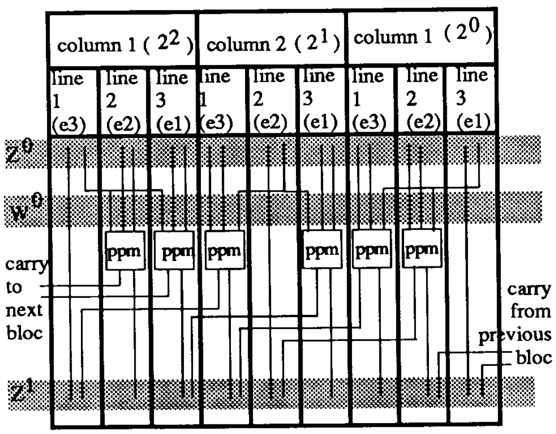


Fig 10: Circuit implementing horizontal and vertical transformations

We use horizontal and vertical transformations in order to build a sequence Z^j which converges to the matrix whose all terms equal 1.

2.5 Examples

i. A parallel adder of real numbers: $n = 1, \ b = 2^{1+1} = 4$

$$\begin{array}{ccc} & Z^j & W^j \\ j=0 & \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \equiv & \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \\ j=1 & \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \equiv & \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \\ j=2 & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \end{array}$$

We obtain the circuit shown *Fig. 11*

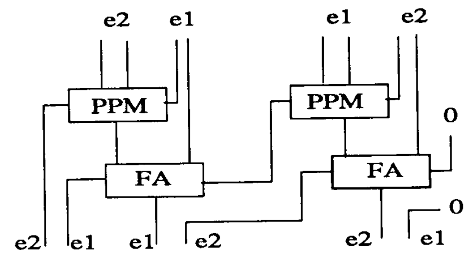


Fig 11: A real redundant adder

ii. A parallel adder of two-dimensional vectors: $n = 2, \ b = 2^3 = 8$

The sequences Z^j and W^j are:

$$\begin{array}{lcc} j=0 & \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \equiv & \begin{pmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix} \\ j=1 & \begin{pmatrix} 2 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 2 \end{pmatrix} \equiv & \begin{pmatrix} 3 & 3 & 0 \\ 0 & 3 & 3 \\ 3 & 0 & 3 \end{pmatrix} \\ j=2 & \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \equiv & \begin{pmatrix} 3 & 0 & 2 \\ 2 & 3 & 0 \\ 0 & 2 & 3 \end{pmatrix} \\ j=3 & \begin{pmatrix} 1 & 0 & 3 \\ 3 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} & \\ j=4 & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} & \end{array}$$

We obtain the circuit presented in figure 8.

iii. A parallel adder of three-dimensional vectors: $n = 3, \ b = 2^4$. The sequences Z^j and W^j are:

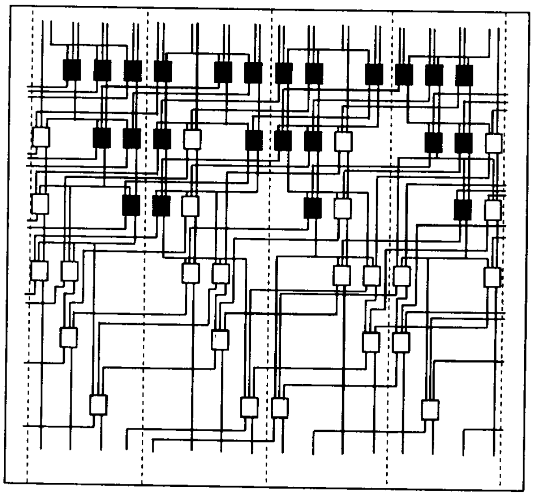
$$\begin{array}{lcc} j=0 & \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix} \equiv & \begin{pmatrix} 1 & 3 & 3 & 3 \\ 3 & 1 & 3 & 3 \\ 3 & 3 & 1 & 3 \\ 3 & 3 & 3 & 1 \end{pmatrix} \\ j=1 & \begin{pmatrix} 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 \\ 2 & 2 & 1 & 2 \end{pmatrix} \equiv & \begin{pmatrix} 3 & 3 & 3 & 0 \\ 0 & 3 & 3 & 3 \\ 3 & 0 & 3 & 3 \\ 3 & 3 & 0 & 3 \end{pmatrix} \\ j=2 & \begin{pmatrix} 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \end{pmatrix} \equiv & \begin{pmatrix} 3 & 3 & 0 & 2 \\ 2 & 3 & 3 & 0 \\ 0 & 2 & 3 & 3 \\ 3 & 0 & 2 & 3 \end{pmatrix} \\ j=3 & \begin{pmatrix} 2 & 1 & 0 & 3 \\ 3 & 2 & 1 & 0 \\ 0 & 3 & 2 & 1 \\ 1 & 0 & 3 & 2 \end{pmatrix} \equiv & \begin{pmatrix} 3 & 0 & 1 & 4 \\ 4 & 3 & 0 & 1 \\ 1 & 4 & 3 & 0 \\ 0 & 1 & 4 & 3 \end{pmatrix} \end{array}$$

$$j = 4 \quad \begin{pmatrix} 1 & 0 & 2 & 3 \\ 3 & 1 & 0 & 2 \\ 2 & 3 & 1 & 0 \\ 0 & 2 & 3 & 1 \end{pmatrix}$$

$$j = 5 \quad \begin{pmatrix} 1 & 0 & 3 & 1 \\ 1 & 1 & 0 & 3 \\ 3 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 \end{pmatrix}$$

$$j = 6 \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

We obtain the circuit presented *fig. 12*.



□ FA Cell ■ PPM Cell

Fig. 12: A 1 radix-16 vectorial digit slice of the adder.

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