# Representation of Numbers in Non-Classical Numeration Systems

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## Abstract

*Numeration systems the basis of which is defined by a linear recurrence with integer coefficients are considered. We give conditions on the recurrence under which the function of normalization which transforms any representation of an integer into the normal one — obtained by the usual algorithm — can be realized by a finite automaton. Addition is a particular case of normalization. The same questions are discussed for the representation of real numbers in basis $\theta$, where $\theta$ is a real number $> 1$. In particular it is shown that if $\theta$ is a Pisot number, then the normalization and the addition in basis $\theta$ are computable by a finite automaton.*

## 1 Introduction

Numbers are used through a symbolic expression and the way they are represented plays an important role in computer science, in arithmetic and in coding theory. The research of numeration systems adequate to specific problems, and in which the arithmetical operations can be accelerated is far from being achieved. The interest for parallel architectures has led to algorithms like the "weak addition" ([1], [12]) where an integer has several representations.

We present here some theoretical results about the possibility of realizing the addition of numbers represented in some non-classical numeration system (extending the usual ones) by means of finite automata.

Finite automata are a "simple" model of computation, since only a finite memory is required. It is known that in the standard $k$-ary numeration system, where $k$ is an integer $\geq 2$, the addition is computable by a finite automaton (*cf* [4]).

In this paper we study numeration systems the basis of which is not a geometric progression but a sequence of integers given by a linear recurrence relation, which paradigm is the sequence of Fibonacci numbers. These numeration systems have also been considered in [5] and [13]. In the Fibonacci numeration system every integer can be represented using digits 0 and 1. The representation is not unique, but one of them is distinguished, the one which does not contain two consecutive 1's (*cf* [15], [11]).

More generally, let $U$ be a strictly increasing sequence of integers such that $1 \in U$. By the greedy algorithm every integer has a representation in basis $U$, that we call the *normal* representation. The *normalization* is the function which transforms any representation on any alphabet onto the normal one. The addition of two integers represented in basis $U$ can be performed that way: just add the two representations digit by digit, without carry, which gives a word on the double alphabet. Then normalize this word to obtain the normal representation of the sum. Thus addition can be viewed as a particular case of normalization.

Our purpose is to study the process of normalization in numeration systems where the basis is defined by a linear recurrence relation with integer coefficients. We call these numeration systems *linear numeration systems*.

In previous works we considered particular cases of linear numeration systems which generalize the Fibonacci numeration system and we showed that normalization is computable by a finite automaton which is obtained by the composition of two sequential machines, one processing words from left to right and the other one from right to left ([6], [7] and [9]). Here we first prove that if the set of normal representations is recognizable by a finite automaton, then the normalization is computable by a finite automaton if and only if the set of words having value 0 in basis $U$ is recognizable by a finite automaton (Proposition 2.1). To every word one associates a polynomial. Then we consider words which can be associated to polynomials belonging to the ideal generated by the characteristic polynomial $P$ of the linear recurrence. Obviously every word of this set is equal to 0 in basis $U$. We give a construction which links recognizability by a finite automaton and division of polynomials by $P$. We prove that the set of words associated to the ideal $(P)$, on any alphabet, is recognizable by a finite automaton if and only if $P$ has no root of modulus 1 (Theorem 2.1). If $P$ has one root of modulus 1, then there exist alphabets on which the normalization is not computable by a finite automaton.

In a similar manner we discuss the representation of real numbers in basis $\theta$ where $\theta$ is a real number $> 1$. The normal $\theta$-representation of a real number is called the *θ-development* or the *θ-expansion* in the literature [14]. The θ-developments of real numbers, when θ is not necessarily an integer, have been used for fast computation of elementary functions (*cf* [12]).

The notion of normalization is defined for the θ-representation as for the integers. If θ is an algebraic integer then a construction similar to the one given for the integers links the recognizability of the set of infinite words equal to 0 to the property of the minimal polynomial of θ of having no root of modulus 1 (Theorem 3.1).

We prove that the normalization is computable by a finite automaton if and only if the set of infinite words equal to 0 in basis θ is recognizable by a finite automaton (Proposition 3.2). Thus the normalization in basis θ is computable by a finite automaton on any alphabet if and only if the minimal polynomial of θ has no root of modulus 1 and if $\sum_{n\geq 0} s_n\theta^{-n} = 0$ implies $\sum_{n\geq 0} s_n\alpha^{-n} = 0$ for every conjugate α of modulus > 1 (Theorem 3.2).

Let θ be an algebraic integer > 1; θ is a *Pisot* number if its conjugates have modulus < 1; θ is a *Salem* number if its conjugates have modulus ≤ 1, and it is not a Pisot number. Thus, if θ is a Pisot number, then the normalization in basis θ is computable by a finite automaton on any alphabet — and addition also. If θ is a Salem number, there exist alphabets on which normalization is not computable by a finite automaton (Corollary 3.1). These results have strong connexion with symbolic dynamics, that we do not discuss here.

The integers and the golden mean $\frac{1+\sqrt{5}}{2}$ being Pisot numbers, our results cover the most standard numeration systems. All proofs can be found in [8].

## 2 The integers

### Representation of integers

Only positive numbers are considered. Let $U = (u_n)_{n\geq 0}$ be a strictly increasing sequence of integers with $u_0 = 1$. Every positive integer $N$ can be written with respect to the basis $U$, *i.e.* it is possible to find $n \geq 0$ and integers $d_0, \cdots, d_n$ such that $N = d_0u_n + \cdots + d_nu_0$ by the following algorithm (folklore):
Given integers $x$ and $y$ let us denote by $q(x, y)$ and $r(x, y)$ the quotient and the remainder of the Euclidean division of $x$ by $y$.
Let $n \geq 0$ such that $u_n \leq N < u_{n+1}$ and let $d_0 = q(N, u_n)$ and $r_0 = r(N, u_n)$, $d_i = q(r_{i-1}, u_{n-i})$ and $r_i = r(r_{i-1}, u_{n-i})$ for $i = 1, \cdots, n$. Then $N = d_0u_n + \cdots + d_nu_0$.

For $0 \leq i \leq n$, $d_i < \frac{u_{n-i+1}}{u_{n-i}}$; thus if the ratio $\frac{u_{n+1}}{u_n}$ is bounded by a positive constant $K$ for all $n \geq 0$ ($K$ minimal), then $0 \leq d_i \leq K-1$. The set $A = \{0, 1, \cdots, K-1\}$ is called the *canonical alphabet* of digits associated to the basis $U$, and $(U, A)$ is the *canonical numeration system* associated to $U$.

The word $d_0 \cdots d_n$ of $A^*$ obtained by this algorithm is called the *normal representation* of the integer $N$ in basis $U$. It is denoted by $< N >= d_0 \cdots d_n$. The normal representation of 0 is the empty word ε.

More generally, a ***numeration system*** is given by a strictly increasing sequence $U = (u_n)_{n\geq 0}$ of positive integers, with $u_0 = 1$, called the ***basis***, and a finite subset $C$ of **N**, the alphabet of ***digits***. A ***representation*** of an integer $N$ in the system $(U, C)$ is a word $d_0 \cdots d_n$ of the free monoid $C^*$ such that $N = d_0u_n + \cdots + d_nu_0$.

The normal representation of an integer $N$ has maximal length among the representations of $N$ not beginning by a 0. It is also the greatest (for the lexicographical ordering) of all the representations of $N$ of this same length in basis $U$. Given $(U, C)$, the mapping $\pi : C^* \to \mathbf{N}$ is defined by $\pi(d_0 \cdots d_n) = d_0u_n + \cdots + d_nu_0$. The ***normalization*** $\nu_C$ is the mapping which associates to a word $f$ of $C^*$ the normal representation of the integer represented by $f$.

The normalization is linked to the problem of addition of two integers written in basis $U$. To add two integers $N$ et $P$ of respective representation $f = f_0 \cdots f_k$ and $g = g_0 \cdots g_j$ in $(U, A)$ we add $f$ and $g$ digit by digit from the right and without carry. Let $f \oplus g = f_0 \cdots f_{k-j-1}(f_{k-j} + g_0) \cdots (f_k + g_j)$ (if $k \geq j$). Then $f \oplus g$ is a word written on the alphabet $\{0, \cdots, 2K-2\}$. The addition of $N$ and $P$ reduces to the normalization of $f \oplus g$.

In this paper we study numeration systems where the basis is defined by

$$u_{n+m} = a_1u_{n+m-1} + \cdots + a_mu_n$$

$$a_i \in \mathbf{Z},\ 1 \leq i \leq m,\ a_m \neq 0.$$

These systems are called ***linear numeration systems***. The ratio $\frac{u_{n+1}}{u_n}$ is bounded for all $n \geq 0$ and the canonical alphabet is included in $\{0, \cdots, K-1\}$ with $K < \max(a_1 + \cdots + a_m, \max\{\frac{u_{i+1}}{u_i} \mid 0 \leq i \leq m-2\})$.
If $m = 1$ and $a_1 \geq 2$ the system is the standard $a_1$-ary numeration system with $A = \{0, \cdots, a_1 - 1\}$ for canonical alphabet.

EXAMPLE **2.1** . — The Fibonacci numeration system $\mathcal{F}$ is defined by the sequence of Fibonacci numbers generated by the linear recurrence $u_{n+2} = u_{n+1} + u_n$, with $u_0 = 1$ and $u_1 = 2$. The canonical alphabet is $\{0, 1\}$. The representations of the integer 24 in $\mathcal{F}$ on $\{0, 1\}$ are the following : 101111, 110011, 110100, 1000011, 1000100. The normal representation of 24 is 1000100. The normal representation of an integer in $\mathcal{F}$ is the one that does not contain two consecutive 1's (*cf* [15]). □

### Normalization of finite words

First let us give some definitions. More details can be found in [4] and [2]. Let $M$ be a monoid. The family Rat$M$ of ***rational*** subsets of $M$ is the least family of subsets of $M$ containing the finite subsets and closed under product, union and the star operation.
A ***finite automaton*** $\mathcal{A} = (E, Q, I, T)$ is a directed graph labelled by letters of the alphabet $E$, with a finite set $Q$ of vertices called ***states***. $I \subset Q$ is the set of ***initial*** states, and $T \subset Q$ is the set of ***terminal*** states. A path in $\mathcal{A}$

is said to be *successful* if it starts in $I$ and terminates in $T$. The set of successful pathes is the *behavior* of $\mathcal{A}$. A word $w$ of $E^*$ is *recognized* by $\mathcal{A}$ if it is the label of a successful path of $\mathcal{A}$. A subset of $E^*$ is *recognizable* if it is the behavior of a finite automaton on $E$. The recognizable subsets of $E^*$ are exactly the rational subsets of $E^*$ by Kleene Theorem (*cf* [4]), and we shall use both denominations.

Let $E$ and $F$ be two alphabets. A *transducer* $\mathcal{T}$ is a finite automaton with edges labelled by couples of $E^* \times F^*$. A relation $R \subset E^* \times F^*$ is *rational* if and only if it is the behavior of a transducer. From now on we shall use the denomination *rational*.

We assume that the characteristic polynomial $P(X) = X^m - a_1X^{m-1} - \cdots - a_m$ of $U$ has a real root $\theta > 1$ which dominates strictly the modulus of its conjugates. $A$ is the canonical alphabet and $L(U) \subset A^*$ is the set of normal representations of the integers in basis $U$.

If $c > 0$ is an integer, let $C = \{0, \cdots, c\}$, $\tilde{C} = \{-c, \cdots, c\}$ and

$$Z(U,c) = \{f = f_0 \cdots f_n \in \tilde{C}^* \mid f_0u_n + \cdots + f_nu_0 = 0\}$$

be the set of words on $\tilde{C}$ equal to 0 in basis $U$.

PROPOSITION **2.1** .— *If the set of normal representations $L(U)$ is rational, then the normalization $\nu_C : C^* \rightarrow A^*$ is a rational function if and only if the set $Z(U,c)$ of words of $\tilde{C}^*$ equal to 0 in basis $U$ is rational.*

To prove that if $\nu_C$ is rational then $Z(U,c)$ is rational, it is necessary to give a precise characterization of the normalization.

Let $E$ and $F$ be two alphabets. The length of a word $f$ is denoted by $|f|$. The set of words on $E$ of length $\leq k$ is denoted by $E^{\leq k}$. Recall that a relation $R \subseteq E^* \times F^*$ is *length-preserving* if, for every $(f,g) \in R$, $|f| = |g|$ (*cf* [4]). This is equivalent to $R \subseteq (E \times F)^*$.

DEFINITION **2.1** .— *A relation $R \subseteq E^* \times F^*$ is said to have* bounded length differences *if there exists $k \in \mathbf{N}$ such that, for every $(f,g) \in R$, $||f| - |g|| \leq k$.*

PROPOSITION **2.2** .— [8], [10] *A rational relation of $E^* \times F^*$ which has length differences bounded by $k$ is equal to the behavior of a transducer $\mathcal{T} = (E \times F, Q, \alpha, T)$ with edges labelled by elements of $E \times F$, equipped with an initial partial function $\alpha : Q \rightarrow (E^{\leq k} \times \varepsilon) \cup (\varepsilon \times F^{\leq k})$.*

The behavior of a transducer of this kind is defined as follows. A couple $(f,g) \in E^* \times F^*$ is recognized by $\mathcal{T}$ if there exist $i \in Q$ and $t \in T$, such that $\alpha(i) = (u,v)$ is defined, $f = uf'$, $g = vg'$ and $(f',g')$ is the label of a path from $i$ to $t$.

Coming back to the linear numeration systems we have

PROPOSITION **2.3** .— *The normalization in basis $U$, restricted to words not beginning by 0, has bounded length differences.*

Define a mapping between words of $\tilde{C}^*$ and polynomials of $\mathbf{Z}[X]$ by :
$f = f_0 \cdots f_n \in \tilde{C}^* \mapsto F(X) = f_0X^n + \cdots + f_n$, $f_i \in \tilde{C}$. The *Gaussian norm* of $F$ is $||F|| = \max_{i=0\cdots n}|f_i|$. This gives a correspondence between words of $\tilde{C}^*$ and polynomials of $\mathbf{Z}[X]$ of norm at most $c$.

Let us denote by $(P)$ the ideal of $\mathbf{Z}[X]$ generated by $P$, and by $I(P,c)$ the trace on $\tilde{C}^*$ of $(P)$, that is $I(P,c) = \{f = f_0 \cdots f_n \in \tilde{C}^* \mid F(X) = f_0X^n + \cdots + f_n \in (P)\}$. This set is strictly included in $Z(U,c)$.

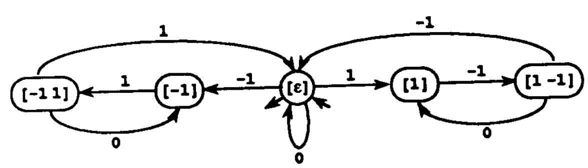
Let $f = uvw$. Then $u$ is a *left factor*, $v$ is a *factor* and $w$ is a *right factor* of $f$. The set of left factors of elements of a language $L$ is denoted by $LF(L)$.

PROPOSITION **2.4** .— *The set $I(P,c)$ is recognizable by a finite automaton if and only if the number of remainders of the Euclidean division by $P$ of polynomials associated to words of $LF(I(P,c))$ is finite.*

Denote by $[f]$ the remainder of the division by $P$ of the polynomial associated to the word $f$. When the number of remainders by $P$ of the words of $LF(I(P,c))$ is finite, the explicit construction of the minimal finite automaton $\mathcal{A} = (\tilde{C}, Q, i, i)$ which recognizes $I(P,c)$ is the following.

(i) the (finite) set of states $Q$ is equal to the set of remainders by $P$ of the elements of $LF(I(P,c))$
(ii) the initial state $i$ is equal to $\{[\varepsilon]\}$
(iii) the terminal state is defined by : $\{[v] \mid v \in I(P,c)\} = i$
(iv) the transitions are of the form $[f] \xrightarrow{a} [fa]$ where $a \in \tilde{C}$.

EXAMPLE **2.2** .— Let $P(X) = X^2 - X - 1$ be the characteristic polynomial of the Fibonacci sequence. The following finite automaton recognizes $I(P,1)$.




□

Since the polynomials considered belong to $\mathbf{Z}[X]$ the number of remainders is finite if and only if the coefficients of the quotient by $F$ are bounded. We thus set the

DEFINITION **2.2** .— *A polynomial $P$ of* $\mathbf{C}[X]$ *satisfies the* bounded division property *(in short (BD)) if, for* *every $c > 0$, there exists a constant $\beta(P, c)$ such that for every polynomial $F$ of $\mathbf{C}[X]$, $F = PQ$, $Q \in \mathbf{C}[X]$, $||F|| \leq c$, implies that $||Q|| \leq \beta(P, c)$.*

PROPOSITION 2.5 . — [3] *The polynomials satisfying the bounded division property are exactly the polynomials having no root of modulus 1.*

From the characterization *supra* we deduce

THEOREM 2.1 . — *The set of words of $\tilde{C}^*$ the associated polynomial of which belongs to $(P)$ is recognizable by a finite automaton for every positive integer $c$ if and only if $P$ has no root of modulus 1.*

EXAMPLE 2.3 . — The Fibonacci polynomial $P(X) = X^2 - X - 1$ has no root of modulus 1, thus $I(P, c)$ is recognizable for every $c \geq 1$. □

EXAMPLE 2.4 . — Let $u_{n+2} = u_{n+1} + 2u_n$ and $P(X) = X^2 - X - 2 = (X+1)(X-2)$ be its characteristic polynomial. One can verify that $I(P, 3) \cap (-1)(3(-3))^*1(3(-3))^*2 = \{(-1)(3(-3))^p1(3(-3))^p2 \mid p \geq 0\}$. Since this set is not rational, $I(P, 3)$ is not rational either. □

We give now a necessary condition for the rationality of the normalization in basis $U$.

THEOREM 2.2 . — *If $P$ has one root of modulus 1, then there exists $c_0 > 0$ such that for every $c \geq c_0$, the normalization $\nu_C$ is not rational.*

The question whether $P$ has no root of modulus 1 implies that the normalization in basis $U$ is rational on any alphabet is still open.

# 3 The real numbers

## Representation of real numbers

Let $\theta > 1$ and $x \geq 0$ be two real numbers. Every infinite sequence of positive integers $(z_n)_{n\geq 0}$ such that $x = \sum_{n\geq 0} z_n\theta^{-n}$ is a *$\theta$-representation* of $x$. A particular $\theta$-representation called the *$\theta$-development* or the *$\theta$-expansion* can be computed by the following algorithm (*cf* [14]).
Denote by $[y]$ and by $\{y\}$ the integer part and the fractional part of a number $y$.
Let $x_0 = [x]$ and $r_0 = \{x\}$, and, for $i \geq 1 : x_i = [\theta r_{i-1}]$ and $r_i = \{\theta r_{i-1}\}$. Then $x = \sum_{k\geq 0} x_k\theta^{-k}$.

For $i \geq 1, x_i < \theta$. If $\theta \in \mathbf{N}$, the canonical alphabet is $A = \{0, \cdots, \theta - 1\}$ and if $\theta \notin \mathbf{N}, A = \{0, \cdots, [\theta]\}$. We write $x = x_0.x_1x_2\cdots$ where $x_0$ is the integer and $.x_1x_2\cdots$ is the fractional part of $x$. The $\theta$-development of $x$ is the normal $\theta$-representation of $x$ and it is greater for the lexicographic ordering than any $\theta$-representation of $x$.

It is clear that if $\theta = t_0.t_1t_2\cdots$ is the $\theta$-development of $\theta$, then $1 = 0.t_0t_1\cdots$. The sequence $t_0t_1\cdots$ is denoted by $d(1)$ and called by extension the $\theta$-development of 1. Let $x \in [0, 1[$ of $\theta$-development $0.x_1x_2\cdots$. The sequence $x_1x_2\cdots \in A^{\mathbf{N}}$ is also said to be the $\theta$-development of $x$.

EXAMPLE 3.1 . — Let $\theta = (1+\sqrt{5})/2$. Then $d(1) = 11$. Let $\theta = (3+\sqrt{5})/2$. Then $d(1) = 21^\omega$. □

## Normalization of infinite words

Let $C$ be any finite subset of integers. As for the integers the *normalization* function $\nu_C : C^{\mathbf{N}} \to A^{\mathbf{N}}$, where $A$ is the canonical alphabet, maps a sequence $(y_n)_n$ of numerical value $x$ in basis $\theta$ onto the $\theta$-development of $x$. We characterize the numbers $\theta$ such that the normalization in basis $\theta$ is rational on any alphabet.

Let us fix some definitions. An infinite path in a finite automaton $\mathcal{A} = (E, Q, I, T)$ is *successful* if it starts in $I$ and goes infinitely often through $T$. The *infinite behavior* of an automaton is the set of all its successful pathes. A subset of $E^{\mathbf{N}}$ is said to be *recognizable* if it is the infinite behavior of a finite automaton, that is if it is Büchi-recognizable (*cf* [4]).
A relation $R \subset E^{\mathbf{N}} \times F^{\mathbf{N}}$ is *rational* if it is the infinite behavior of a transducer.

As for the integers we first consider the set of infinite words on $\tilde{C}^{\mathbf{N}}$ equal to 0 in basis $\theta$, $Z(\theta, c) = \{s = (s_n)_{n\geq 0} \in \tilde{C}^{\mathbf{N}} \mid \sum_{n\geq 0} s_n\theta^{-n} = 0\}$.

To every infinite word $s = (s_n)_{n\geq 0}$ of $\tilde{C}^{\mathbf{N}}$ is associated a formal power series $S(X) = \sum_{n\geq 0} s_nX^n$ in $\mathbf{Z}[[X]]$ which *Gaussian norm* is $||S|| = \sup_{n\geq 0} |s_n| \leq c$.

One can show that it is not a restriction to suppose that $\theta$ is an algebraic integer. A construction similar to the one given in Section 2 links the recognizability of $Z(\theta, c)$ and the division of polynomials by the minimal polynomial $M$ of $\theta$. Let us denote by $LF(Z(\theta, c))$ the set $\{w \in \tilde{C}^* \mid \exists s \in \tilde{C}^{\mathbf{N}},\ ws \in Z(\theta, c)\}$.

PROPOSITION 3.1 . — *Let $\theta$ be an algebraic integer $> 1$. The set $Z(\theta, c)$ is recognizable by a finite automaton if and only if the number of remainders of the division by the minimal polynomial $M$ of $\theta$ of polynomials associated to words of $LF(Z(\theta, c))$ is finite.*

EXAMPLE 3.2 . — The Fibonacci polynomial $P(X) = X^2 - X - 1$ is the minimal polynomial of $\theta = (1+\sqrt{5})/2$. The finite automaton constructed in Example 2.2, with every state terminal, has for infinite behavior the set of infinite words on $\{-1, 0, 1\}$ equal to 0 in Fibonacci basis $(1+\sqrt{5})/2$. □

As above, the number of remainders is finite if and only if the coefficients of the quotient of the division are bounded since the polynomials belong to $\mathbf{Z}[X]$. With a result similar to the one expressed in Proposition 2.5, we prove that.

THEOREM 3.1 . — *Let $\theta$ be an algebraic integer $> 1$, $M$ its minimal polynomial. The set $Z(\theta, c)$ is recognizable for every $c$ if and only if $M$ has no root of modulus 1, and if for every infinite word $s = (s_n)_{n\geq 0}$ of $Z(\theta, c)$, one has $\sum_{n\geq 0} s_n \alpha^{-n} = 0$ for every root $\alpha$ of modulus $> 1$ of $M$.*

Using the same tools as in Proposition 2.1 we are able to show

PROPOSITION 3.2 . — *The normalization $\nu_C : C^{\mathbf{N}} \rightarrow A^{\mathbf{N}}$ is rational if and only if $Z(\theta, c)$ is recognizable.*

The proof uses the following property of the normalization (*cf* [10]).

PROPOSITION 3.3 . — *If the normalization in basis $\theta$ is rational, it has a bounded delay.*

The previous results can be put together into the following statement.

THEOREM 3.2 . — *The normalization $\nu_C$ in basis $\theta$ is rational on any alphabet $C$ if and only if the minimal polynomial of $\theta$ has no root of modulus 1 and if $|s_n| \leq c$, $\sum_{n\geq 0} s_n \theta^{-n} = 0$ implies $\sum_{n\geq 0} s_n \alpha^{-n} = 0$ for every conjugate $\alpha$ of modulus $> 1$.*

COROLLARY 3.1 . — *Let $\theta$ be a Pisot number. For every alphabet $C$, the normalization $\nu_C$ in basis $\theta$ is rational (and in particular the addition also).*
*Let $\theta$ be a Salem number. There exists an integer $c_0$ such that for every integer $c \geq c_0$ the normalization $\nu_C$ in basis $\theta$ is not rational.*

EXAMPLE 3.3 . — Let $\theta = (1 + \sqrt{5})/2$. Then $\theta$ is a Pisot number, and the normalization is rational on any alphabet. □

EXAMPLE 3.4 . — Let $\theta = (3 + \sqrt{5})/2$. The minimal polynomial of $\theta$ is $X^2 - 3X + 1$ and $\theta$ is a Pisot number. The normalization is rational on any alphabet. □

EXAMPLE 3.5 . — Let $\theta$ be the dominant root of the polynomial $X^4 - 2X^3 - 2X^2 - 2X + 1$. $\theta$ is a Salem number and $d(1) = 2(211)^{\omega}$. There exists $c_0$ such that for every $c \geq c_0$ the normalization on $C$ is not rational. □

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