Generation and Analysis of Hard to Round Cases for Binary Floating Point Division

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Abstract

We investigate two sets of hard to round p×p bit fractions arising from division of a normalized p bit floating point dividend by a normalized p bit floating point divisor. These sets are characterized by the p×p bit fraction's quotient bit string, beginning with or just after the round bit, having the maximum number (p-1) of repeating like bits, specifically 00...01 or 11...10 for the directed rounding "RD-hard" set and 100...01 or 011...10 for the round-to-nearest "RN-hard" set. We show both the p×p bit RD-hard and RN-hard sets to be of size at least 2^{p-2} and at most 2^{p-1}. Two dimensional quotient vs. divisor plots empirically reveal both the RD-hard and RN-hard sets of p×p bit fractions to be jointly widely distributed. Analysis of patterns and linear sequences of fractions visible in the quotient vs. divisor plots leads to simplified procedures for generating test suites of hard to round fractions. Our strongest computational result is the derivation of formulas that allow $2^{(p/2)+O(1)}$ RD-hard and RN-hard p×p bit fractions to be enumerated based on sequential incrementation of respective numerators and denominators.

1 Introduction and Summary

Various procedures for generating hard to round test cases for verifying implementations of arithmetic operations and functions have been described and analyzed in numerous studies [Ka87, Ta89, SS93, Pa99, LM99, IM99, MM00]. Test cases for floating point division and square root are of particular interest since the IEEE standard [IEEE] mandates that conforming implementations must correctly round all results by both directed and nearest modes. Many division and square root algorithms in current hardware designs employ lookup tables and iterative algorithms generating super precise approximations before rounding. Algorithms for rounding sufficiently accurate approximations for division, square root, and square root reciprocal have been described and investigated [CG99, IM99, Ka87]. Discussions of tests discovering actual chip design shortcomings in implementations of these instructions have appeared in [Pa99, IM00].

Test suites for division should be well distributed by divisor value to cover all values in a divisor reciprocal table, and well distributed by quotient value to test output convergence and rounding algorithms over a wide range of output values. In this paper we focus solely on hard to round test cases for division,

In Section 2 we develop our *p*×*p* bit fraction model of binary floating point division. The hardest to round cases of *p*×*p* bit division are first characterized by four extreme length quotient *p*-bit string patterns commencing with the round bit: 00...01, 011...1, 100...0, and 11...10. Then an equivalent characterization employing fractions is given identifying those *p*×*p* bit hard to round fractions $\frac{n}{d}$ near endpoints $\frac{n'}{2^{p-1}}$ or midpoints $\frac{n''}{2^p}$ that give rise to the extreme binary quotient bit string patterns. Finding such a hard to round fraction directly identifies an input dividend and divisor pair for inclusion in a division test suite.

In Section 3 we employ and extend tools from the foundations of fractions and continued fractions that shows it is possible to exhaustively compute all hard to round fractions at an effective cost of about one *p*×*p* bit GCD operation per *p*×*p* fraction. Our principal result in Section 3 is the establishment of several symmetries in the exhaustive 2-dimensional quotient vs divisor plots of hard to round fractions that allows simple determination of from 1 to 5 other hard to round fractions for each new hard to round fraction computed with a GCD.

Our main result in Section 4 is several formulas for enumeration of sequences of $2^{(p/2)+O(1)}$ hard to round fractions that align in a linear progression in the quotient vs. divisor plots. We provide algorithmic code segments indicating how these linear progressions of fractions may be efficiently generated by simple incrementation of respective numerators and denominators. Several test suites containing hundreds of millions of RD-hard and RN-hard double precision (53 bit) dividend, divisor pairs are identified by this process. We further note that hard to round fractions may be ranked by their distance to the closest endpoint or midpoint. Using only the desired rank and precision as input, we show how the associated hard to round fraction may be found by a *p*×*p* GCD computation.

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2 Hard to Round $p \times p$ Bit Fractions

A positive *p-bit number* is a binary rational with a unique factorization $z = 2^e i,\ 1 \le i \le 2^p - 1,\ i$ odd. For $e \ge 0, z = 2^e i$ is a *p-bit integer*. For $e < 0, z = i/2^{|e|}$ is a non integral binary rational. The *precision* $p \ge 1$ provides a measure of the size of the p-bit number's significand bit string in the normalized floating point factored format $z = 2^{e'}(1.b_1 b_2 \cdots b_{p-1})$.

A $p \times p$ *bit fraction* denotes a fraction $\frac{n}{d}$ where the numerator and denominator are positive p-bit integers. The rational value $q = \frac{n}{d}$ is termed the infinitely precise $p \times p$ *bit quotient*. For our purposes herein a *normalized* $p \times p$ bit fraction has the denominator in the range $1 \le d \le 2^p - 1$, and the numerator in the range $d \le n < 2d$, so then the *normalized* $p \times p$ bit quotient is in the standard binade $1 \le q = \frac{n}{d} = 1.b_1 b_2 \cdots < 2$. Note that if this normalization yields a numerator greater than 2^p, it must be even.

Rounded p-bit floating point division may be visualized by enumerating the sequence, $Q_{p \times p}$, of irreducible normalized $p \times p$ bit fractions ordered by their rational values. More particularly, this sequence may be enumerated separately over each ulp interval $[\frac{i}{2^{p-1}}, \frac{i+1}{2^{p-1}}]$, for $2^{p-1} \le i \le 2^{p-1} - 1$, to investigate the effect of rounding a quotient to one or the other endpoint. Figure 1 illustrates the sequence $Q_{5 \times 5}$.

Each of the 16 rows of Figure 1 corresponds to a 5-bit one ulp interval and contains the 5×5 bit fractions having the same initial 5-bit quotient string $1.b_1 b_2 b_3 b_4$, with their position left to right scaled by the value of their tails $.b_5 b_6 \cdots$. The sets of rounding equivalent fractions for determining rounded p-bit quotients are readily identified by their further partition about the midpoint line corresponding to irreducible fractions of the form $\frac{n}{2^p}$. The collective values of the tails $.b_5 b_6 \cdots$ for all 5×5 bit quotients have been illustrated in Figure 1 by projecting each 5×5 bit fraction value to a tic mark on the one ulp interval at the bottom of the figure. The tail value projections appear non uniform with interesting gaps about the midpoint and at both ends of the ulp interval.

The spacing between successive fractions of $Q_{5 \times 5}$ in Figure 1 appears quite erratic but can be explained by reference to the component denominators and numerators. Successive fractions have differences of the form $\frac{n}{d} - \frac{n'}{d'} = \frac{nd' - n'd}{dd'} = \frac{i}{dd'}$. In [MM00] we showed for enumerated $Q_{p \times p}$ that $|i|$=1 unless n, n' are both even, in which case $|i|$=2. This relationship explains both the variable spacing and the existence of wider gaps noted on ei-

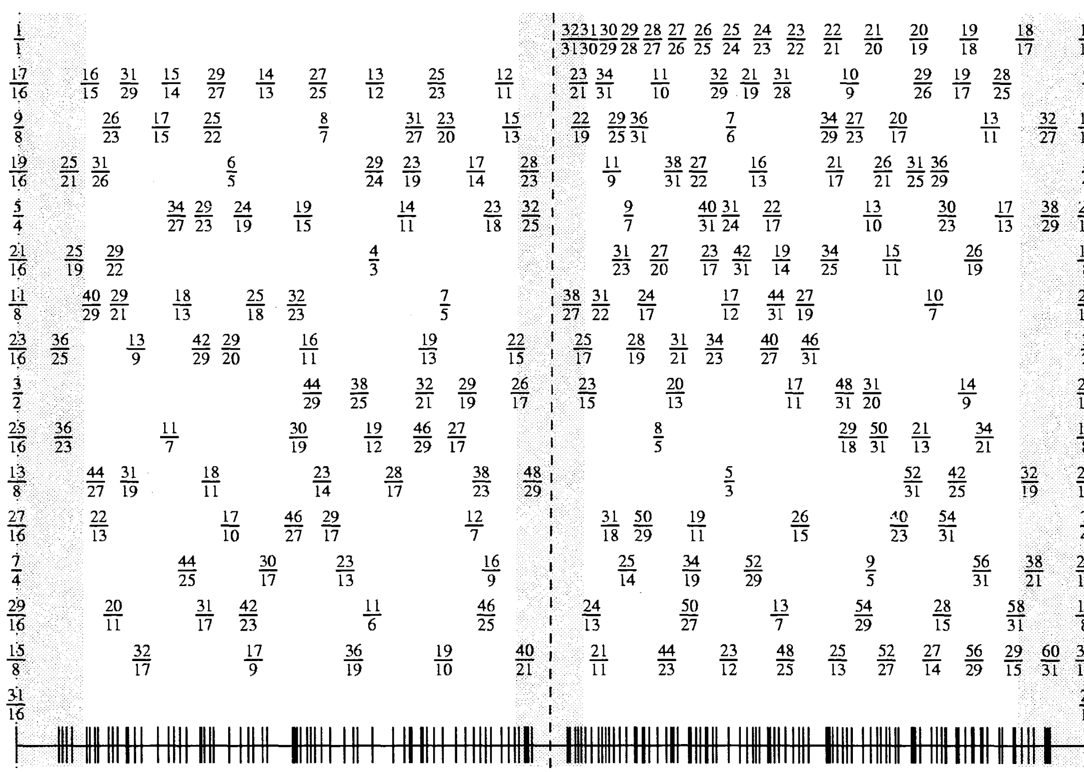

Figure 1: A scaled plot of the 5×5 bit fractions $Q_{5 \times 5}$ partitioned into 5-bit ulp intervals

ther side of a fraction having a relatively small denominator, such as $\frac{4}{3}$ or $\frac{7}{5}$. This further explains why no $p{\times}p$ bit fraction can be closer than $\frac{1}{2^{2p}}$ to the midpoint in such a projection, and why it can be no closer than $\frac{1}{2^{2p-1}}$ to an endpoint unless it is one of the exact endpoints $\frac{n}{2^{p-1}}$.

It is well known that the leading p bits of the tail of a $p{\times}p$ bit quotient commencing with the round bit can never be all 1's, and is all 0's only when it is the value of a $p{\times}p$ bit fraction that is a *p-bit endpoint* $\frac{n}{2^{p-1}}, 2^{p-1} \le n < 2^p - 1$. Furthermore a string of p-1 like bits following and complementary to the round bit is the longest such string for any $p{\times}p$ bit quotient. Thus the four leading p-bit strings 00...01, 011...1, 100...0, 11...10 of the tail constitute the extreme case run lengths of like bits commencing with or after the round bit of a $p{\times}p$ bit quotient. $p{\times}p$ bit fractions yielding such results are of considerable interest for testing division algorithms as they identify input dividend, divisor pairs giving rise to the most sensitive rounding boundary output cases.

A normalized $p{\times}p$ bit quotient is termed *hard to round for directed rounding (RD-hard)* if the leading p-bits of the tail commencing with the round bit is either 00...01 or 11...10, and is *hard to round to nearest (RN-hard)* if this p-bit segment is either 011...1 or 100...0.

Equivalently, in the language of normalized $p{\times}p$ bit fractions, $\frac{n}{d}$ is: (i) RD-hard iff there is an endpoint fraction $1 \le \frac{n'}{2^{p-1}} \le 2$ such that $0 < \left| \frac{n'}{2^{p-1}} - \frac{n}{d} \right| < \frac{1}{2^{2(p-1)}}$, (ii) RN-hard iff there is a *midpoint* fraction $1 < \frac{n''}{2^p} < 2$ such that $\left| \frac{n''}{2^p} - \frac{n}{d} \right| < \frac{1}{2^{2p-1}}$. Figure 1 is enhanced by shaded regions at both ends and in the middle to highlight those $p{\times}p$ bit fractions that satisfy the hard to round criteria.

3 Distribution of Hard to Round Fractions

For division algorithm test suites it is desirable to identify a large sample of hard to round fractions distributed reasonably uniformly by divisor value so that all reciprocal approximations from a lookup table will be tested. It is similarly useful for the suite to be distributed reasonably uniformly by quotient values for convergence algorithms.

It is informative to visualize certain sets of RD-hard and RN-hard fractions by plotting the fractions on a grid with the x axis representing a denominator value from 2^{p-1} to 2^p and the y axis representing a quotient value between 1 and 2. Figure 2 shows such *QD-plots* for all RD-hard fractions for p=7 and p=13. For p=7, the 42 fractions themselves are plotted, and for p=13, the 2800 fractions are represented by dots.

Note that the numerators in QD-plots increase on any ray away from the origin, with the hyperbola $n = qd = 2^p$ denoting the boundary beyond which all numerators must be even to be p bit integers. The spatial distribution appears substantially uniform on either side of the hyperbola for p=13, which is encouraging regarding the development of test suites. In addition, there appears to be considerable diagonal axes symmetry and certain embedded patterns for these RD-hard $p{\times}p$ bit fractions.

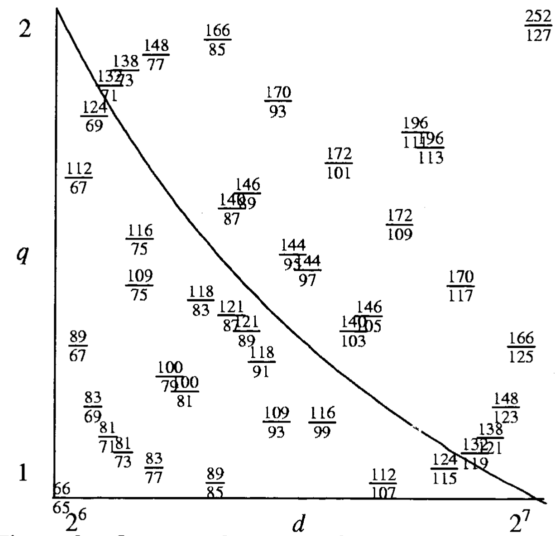

Figure 2a: QD-plot of RD-hard fractions for p=7

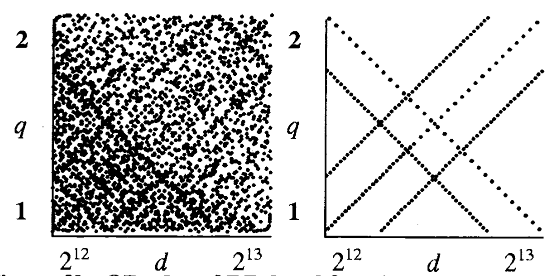

Figure 2b: QD-plot of RD-hard fractions for p=13

Some 296 of the 2800 fractions in the QD-plot for p=13 fall essentially on just 5 diagonal lines as separately illustrated in Figure 2b. We will show in Section 4 how these fractions may be simply and efficiently generated.

Figure 3 shows a corresponding QD-plot for RN-hard fractions for p=13. While the number and overall distribution of RN-hard cases appears substantially similar to the corresponding RD-hard cases, the axial symmetry is limited and the "diagonal" lines are less visible and flattened to one half the slope compared to the RD-hard plot. Some 290 of the 2832 RN-hard fractions for $p=13$ are separately shown to fall essentially on 8 flattened diagonal lines in Figure 3. The RD-hard and RN-hard fractions on the illustrated lines of Figures 2b and 3, respectively, are related in a manner that allows their efficient joint generation and enumeration, as will be shown in Section 4.

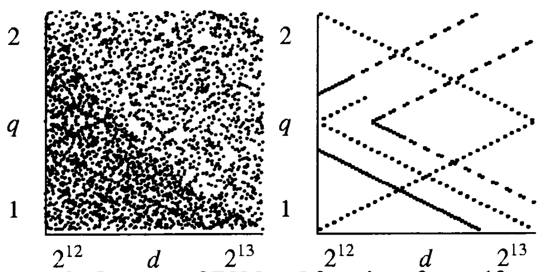

Figure 3: QD-plot of RN-hard fractions for p=13

Determination of RD-hard and RN-hard fractions and their associated properties derives from fundamental results on fractions and continued fractions. The following terms and results are readily obtained. The development is similar to the treatment in [MM00], derived from classical treatments in [HW79, ChsIII and X, Kh35], and from the study of fractions as a foundation for finite precision rational arithmetic [MK80].

Terminology and Selected Properties of Fractions

adjacency: $\frac{i}{j}$ *adj* $\frac{k}{l}$, a symmetric relation between fractions denoting $|kj - il| = 1$.

mediant $(\frac{i}{j}, \frac{k}{l})$: an operation on two fractions determining the fraction $\frac{i+k}{j+l}$, termed the *mediant* of $\frac{i}{j}, \frac{k}{l}$.

- The adjacent fractions $\frac{n'}{d'}$ and $\frac{n}{d}$ are at a distance $\frac{1}{d'd}$.
- Every irreducible fraction $\frac{n}{d} \neq \frac{0}{1}, \frac{1}{0}, \frac{1}{1}$ is the mediant of two simpler adjacent fractions.
- The mediant $\frac{n'+n}{d'+d}$ of the adjacent fractions $\frac{n'}{d'} < \frac{n}{d}$ is adjacent to both and falls between them in numeric order, so $\frac{n'}{d'} < \frac{n'+n}{d'+d} < \frac{n}{d}$.
- Every normalized $p \times p$ bit irreducible fraction $1 < \frac{n}{d} < 2$ has a unique continued fraction expansion with at least three partial quotients, and where the final partial quotient is unity.

For example, $$\frac{45}{32} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{1}{1}}}}$$

The *convergents*, $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{38}{27}, \frac{45}{32} = \frac{n}{d}$ are the sequence of irreducible fractions obtained by successively truncating the continued fraction after each partial quotient. The last two convergents $\frac{n''}{d''}, \frac{n'}{d'}$, to $\frac{n}{d}$ are the *parents* of $\frac{n}{d}$ with the last convergent $\frac{n'}{d'}$ being the *closest parent*. The parents can be shown to be uniquely characterized by the fact that they are adjacent fractions having as their mediant the fraction $\frac{n}{d}$. In the example $\frac{n''}{d''} = \frac{7}{5}$ is adjacent to $\frac{n'}{d'} = \frac{38}{27}$ with mediant $\frac{7+38}{5+25} = \frac{45}{32}$. Further note that the closest parent $\frac{38}{27}$ is at a distance $|\frac{45}{32} - \frac{38}{27}| = \frac{1}{32 \times 27} < \frac{1}{2^9}$ and is then RN-hard. Note that the parents of an irreducible binary fraction, $\frac{n}{2^k}$ must both have odd denominators, with one numerator even and one odd. We thus can define for every irreducible normalized binary fraction $1 < \frac{n}{2^k} < 2$:

odd parent: The fraction determining the unique parent of $\frac{n}{2^k}$ with an odd numerator.
even parent: The fraction determining the unique parent of $\frac{n}{2^k}$ with an even numerator.
odd child: The mediant of the even parent and $\frac{n}{2^k}$.
even child: The mediant of the odd parent and $\frac{n}{2^k}$.

Figure 4 illustrates the parents and children of the fractions $\frac{21}{16}$ and $\frac{45}{32}$, with left to right order indicating the numerical order by quotient value, and mediants below parents. The lines of Figure 4 denote adjacent fractions, with the solid lines indicating the closest parent. Note that the closest parent of $\frac{n}{2^k}$ may be the even or odd parent and may be smaller or larger in numerical order.

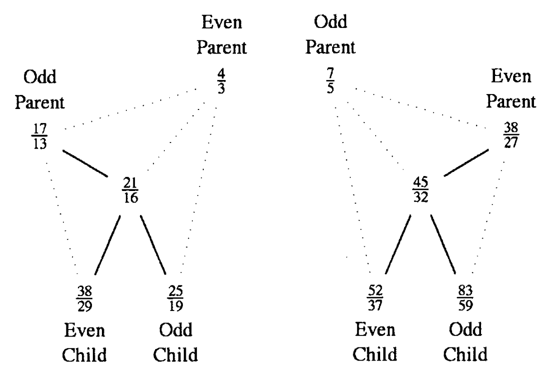

Figure 4: The parents and children of an endpoint $\frac{21}{16}$ and a midpoint $\frac{45}{32}$ of the 5×5 bit fractions

In general, the parity and range of the numerators and denominators of the parents and children of the irreducible fractions $1 < \frac{n}{2^k} < 2$ provide the keys for characterizing the nature of, and analyzing the distribution of, all hard to round fractions for all precisions.

Observation 1: The denominators of the children of the irreducible p bit endpoint $1 < \frac{n}{2^{p-1}} < 2$ are both odd and in the range $[2^{p-1}+1, 2^p-1]$. The denominator of the closest parent of the irreducible p bit midpoint $1 < \frac{n}{2^p} < 2$ is odd and in the range $[2^{p-1}+1, 2^p-1]$. □

Theorem 2: The fraction $1 < \frac{n}{d} < 2$ is an RD-hard $p \times p$ bit fraction iff it is either:

(i) the even child of an irreducible endpoint $1 < \frac{n'}{2^{p-1}} < 2$, or
(ii) the odd child of an irreducible endpoint $1 < \frac{n'}{2^{p-1}} < 2$ with $n < 2^p - 1$. □

Corollary 3: Any interval $1 \le a < b < 2$ of width at least $k2^{p-1}$ must contain the values of at least $\lfloor \frac{k-1}{2} \rfloor$ RD-hard $p \times p$ bit fractions. The total number of RD-hard $p \times p$ bit fractions is at least 2^{p-2} and at most 2^{p-1}. □

Theorem 4: The fraction $1 < \frac{n}{d} < 2$ is an RN-hard $p \times p$ bit fraction iff it is the closest parent of an irreducible midpoint $1 < \frac{n'}{2^p} < 2$ and either n is even or $n \le 2^{p-1}$. □

Corollary 5: Either the irreducible midpoint $1 < \frac{n}{2^p} < 2$ or its *3's complement* irreducible midpoint $\frac{n'}{2^p} = \frac{3 \times 2^p - n}{2^p}$ has a closest parent that is an RN-hard $p \times p$ bit fraction.
Proof: The 3's complement of the closest parent of $\frac{n}{2^p}$ is the closest parent of the 3's complement of $\frac{n}{2^p}$, and one of these closest parents is an even parent. □

Thus the number of RN-hard $p \times p$ bit fractions falls between 2^{p-2} and 2^{p-1}, with exactly 2^{p-2} RN-hard fractions having an even numerator. This result does not rule out large quotient intervals without an RN-hard fraction, but the empirically computed distributions suggest considerable uniformity. The distribution by denominator value is guaranteed by the following result (see [MM00]).

Lemma 6: For every odd denominator $2^{p-1} \le d \le 2^p - 1$, there is an RN-hard $p \times p$ bit fraction $1 < \frac{n}{d} < 2$. □

Computationally the Euclidean GCD algorithm can be used to determine the continued fraction for any irreducible endpoint $1 < \frac{n}{2^{p-1}} < 2$ or irreducible midpoint $1 < \frac{n}{2^p} < 2$. Thus the $p \times p$ bit RD-hard and RN-hard fractions can be found exhaustively employing some $2^{p-1+O(1)}$ GCD computations. They may be generated in quotient order or randomly for testing. Exhaustive determination of all RD-hard and RN-hard fractions for single precision (p=24) is quite tractable but is too time consuming for double precision (p=53). For each hard to round $p \times p$ bit fraction discovered, certain symmetries in the QD-plots can be employed to find other hard to round $p \times p$ bit fractions.

This is practically useful when hard-to-round fractions are generated by a GCD computation on-the-fly for testing a division operation. Deriving a number of hard-to-round fractions from each GCD computation may avoid having the GCD time significantly dominate the division test time.

Symmetries in RD-hard and RN-hard QD-plots

There are three symmetries between RD-hard $p \times p$ bit fractions that serve to characterize and guarantee spatial distribution for RD-hard quotient points in the QD-plot. These symmetries are termed *diagonal*, *sibling*, and *3's complement*, where the latter is dependent on the first two.

Diagonal Symmetry of RD-hard Fractions

This symmetry pairs off all RD-hard fractions about the main diagonal of the QD-plot.

Observation 7: Let $\frac{n}{j}$ adjacent to $\frac{i}{2^{p-1}}$ be RD-hard. Then $\frac{n}{i}$ is adjacent to $\frac{j}{2^{p-1}}$, and is also RD-hard, with $\frac{n}{j}, \frac{n}{i}$ termed a *diagonal* RD-hard pair. □

For the diagonal RD-hard pair $\frac{n}{j}, \frac{n}{i}$ in the QD-plot, first note that the points $(\frac{i}{2^{p-1}}, j)$ and $(\frac{j}{2^{p-1}}, i)$ are precisely symmetric across the main diagonal. Since the plotted $(\frac{n}{j}, j)$ is close to $(\frac{i}{2^{p-1}}, j)$, and $(\frac{n}{i}, i)$ is close to $(\frac{j}{2^{p-1}}, i)$, the RD-hard diagonal pair will appear symmetric for sufficiently large p. This symmetry across the main diagonal is clearly visible in the RD-hard QD-plots for p=7 and 13 in Figure 2.

Theorem 2 guaranteed that for every odd numerator $2^{p-1}+1 \le n \le 2^p - 1$, there is an RD-hard child $\frac{n_c}{d_c}$ adjacent to $\frac{n}{2^{p-1}}$. From Observation 7 $\frac{n_c}{n}$ is then also RD-hard, and we obtain the following guarantee of RD-hard fractions by denominator value.

Observation 8: For every odd denominator $2^{p-1}+1 \le d \le 2^p-1$, there is an RD-hard fraction $\frac{n}{d}$ for some $1 < \frac{n}{d} < 2$. □

Sibling Symmetry of RD-hard Fractions

This symmetry pairs off the odd and even children when they are both RD-hard as in the example of Figure 4. The *sibling RD-hard pair* $\frac{n'}{d'} < \frac{n''}{d''}$ have nearly equal quotient values and denominators summing to $3{\times}2^{p-1}$. Thus the plotted points $(\frac{n'}{d'}, d')$, and $(\frac{n''}{d''}, d'')$, appear symmetric across the vertical axis $d = \frac{3}{2}2^{p-1}$ for sufficiently large p, as evident in Figure 2.

Each sibling RD-hard pair identifies an ordered triple $\frac{n'}{d'} < \frac{n}{2^{p-1}} < \frac{n''}{d''}$ where the binary expansions yield round and sticky bit pairs 11, 00, 01, respectively. All three infinitely precise quotients fall within an interval of width $|\frac{n''}{d''} - \frac{n'}{d'}| = \frac{3}{d''d'} < \frac{3}{2^{2p-1}}$. The latter bound is obtained since $d' = 3{\times}2^{p-1} - d''$ implies $d''d' > 2^{2p-1}$. Thus the three distinct $p{\times}p$ bit quotients for such a triple fall in a tiny sliver of width just 3 parts in 2^p of one ulp. These triples associated with sibling RD-hard pairs provide seemingly challenging examples for division algorithms suitable for inclusion in test suites. For odd precisions, such as $p = 53$ for IEEE double precision, we shall show an efficient procedure for generating a large set of these triples widely distributed by denominator and quotient value in the next section.

3's Complement Symmetry of RD-hard Fractions

This symmetry pairs off an RD-hard $\frac{n}{d}$ with its 3's complement $\frac{n'}{d} = 3 - \frac{n}{d}$.

Observation 9: Let $\frac{n'}{d}$ be RD-hard with n' odd. Then the 3's complement $\frac{n''}{d} = \frac{3d-n'}{d}$ has n'' even and is RD-hard, with $\frac{n'}{d}$, $\frac{n''}{d}$, forming a *3's complement RD-hard pair.* □

With $\frac{n'}{d}$ adjacent to $\frac{n}{2^{p-1}}$, it is straightforward to show $\frac{n''}{d}$ is adjacent to $\frac{3\times 2^{p-1}-n}{2^{p-1}}$ justifying Observation 9. Since $\frac{n'}{d}$ and $\frac{n''}{d}$ have the same denominator with quotient values equidistant from $q = \frac{3}{2}$, there is precise symmetry about the line $q = \frac{3}{2}$ for such pairs.

Note that the 3's complement symmetry follows from the other two symmetries. Let each member of a sibling RD-hard pair be associated with a diagonal RD-hard fraction, and the two such associated fractions together then form a 3's complement RD-hard pair. The stronger result is that the 3's complement pair exhibits precise symmetry.

The distribution and symmetries of the set of all $p{\times}p$ bit RD-hard fractions is best described by dividing the QD-plot range into 9 regions as illustrated in Figure 5.

Regions 4, 5, and 6 fall below and to the left of the hyperbola traced by the curve $\frac{2^p}{x}$ as x goes from 2^{p-1} continuously to 2^p. All RD-hard fractions with odd numerators occur in the regions 4, 5, and 6. Region 6 exhibits the precise 3's complement symmetry for all its member points, and region 4 exhibits the sibling symmetry about $d = \frac{3}{2}2^{p-1}$ for all its member points (refer to Figure 2 for particular examples of these regions for p=7, 13). Only the odd valued numerator fractions of region 4 yield 3's complement RD-hard fractions in region 8, with no symmetric pair about $d = \frac{3}{2}2^{p-1}$ in region 8. The same observation applies between regions 2 and 6. These observations further imply that the number of RD-hard fractions in regions 4 and 6 are the same, with half as many each in regions 2 and 8.

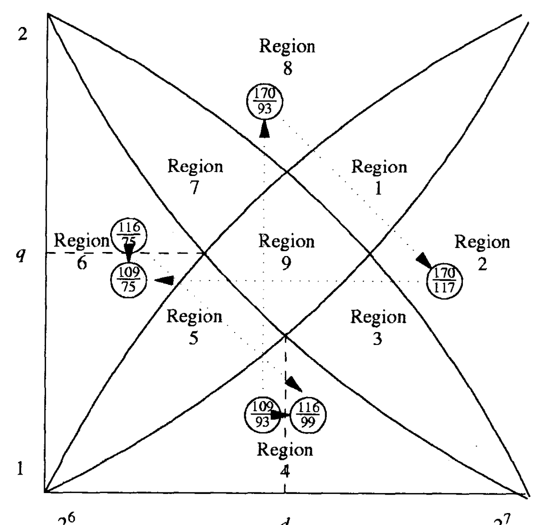

Figure 5: Symmetric Regions of the QD-plot

Within each of the regions 1,3,5,7,9 defined by the reflected hyperbolas, there is diagonal symmetry. Regions 1,5,9 have symmetry about the principal diagonal, and regions 3,7,9 about the cross diagonal. Symmetric reflection between 1,3,5, and 7 is more complex depending on the numerator parity of the RD-hard fractions. An odd numerator point in region 5 is reflected to even numerator points in both regions 3 and 7. An even numerator point in region 5 is reflected about the cross diagonal to another even numerator point in region 1. The number of RD-hard fractions in regions 3 and 7 are equal and the number in region 5 is the sum of the numbers in regions 1 and 3.

For an RD-hard sibling pair $\frac{109}{93}$, $\frac{116}{99}$ in Figure 5's region 4, we trace the symmetries to identify a six tuple $\frac{109}{93} \to \frac{170}{93} \to \frac{170}{117} \to \frac{109}{75} \to \frac{116}{75} \to \frac{116}{99}$ of associated RD-hard fractions that must be present as a regionally distributed subset of RD-hard fractions. Except for a couple of special cases, an RD-hard fraction in any of regions 1,3,5,7,9 can be traced by the symmetries to reveal subsets of four or six RD-hard fractions.

3's Complement Symmetry of RN-hard Fractions

The single symmetry readily apparent for RN-hard fractions is the 3's complement form.

Observation 10: Let $1 < \frac{n}{d} < 2$ be RN-hard with n odd. Then the 3's complement $\frac{n'}{d} = \frac{3d-n'}{d}$ is RN-hard, with $\frac{n}{d}$, $\frac{n'}{d}$ forming a *3's complement RN-hard pair.* □

The RN-hard 3's complement pairs have precise symmetry about the horizontal line $q = \frac{3}{2}$.

For the region 6 of the QD-plot of Figure 5, all RN-hard associated points will exhibit this symmetry as can be seen in Figure 3 for p=13. Further results are needed to better understand the RN-hard distribution throughout the QD-plot.

4 Efficient Generation of Hard to Round Fractions

The QD-plot for RD-hard fractions has features which can be exploited for efficient generation of representative hard to round fractions. The diagonal extending from the origin with q proportional to d spans the full range of q and the full range of d.

Lemma 11: For any odd precision $p \geq 5$, and any odd $k < (\sqrt{2}-1)2^{(p-1)/2}$, let $n_k = (2^{\frac{p-1}{2}} + k)^2$ and $d_k = 2^{p-1} + k2^{\frac{p-1}{2}} - 1$. Then

(i) $\frac{n_k}{d_k}$ is RD-hard and the odd child of $\frac{d_k+2}{2^{p-1}}$,

(ii) $\frac{3(d_k+2)-n_k}{3\times 2^{p-1}-d_k}$ is RD-hard and the even child of $\frac{d_k+2}{2^{p-1}}$.

Proof: Since $|(d_k+2) * d_k - 2^{p-1} * n_k| = 1$, $\frac{n_k}{d_k}$ is adjacent to $\frac{d_k+2}{2^{p-1}}$. Since $d_k > 2^{p-1}$, $\frac{n_k}{d_k}$ is a child of $\frac{d_k+2}{2^{p-1}}$. Furthermore, since k is odd, $\frac{n_k}{d_k}$ is the odd child of $\frac{d_k+2}{2^{p-1}}$. The even child follows from sibling symmetry. □

Referring back to Figure 2 with p = 7, the sibling RD-hard pair $\frac{81}{71}$, $\frac{138}{121}$ corresponds to k = 1 and the pair $\frac{121}{87}$, $\frac{146}{105}$ corresponds to k = 3. Generalizing the formula to even k and including RD-hard fractions that can be determined by the available symmetries leads to a test suite containing some $1.414 \times 2^{(p-1)/2}$ RD-hard fractions distributed virtually linearly on the main diagonal and $2^{(p-1)/2}$ RD-hard fractions distributed virtually linearly on the orthogonal diagonal. These fractions may be computed by simple increments of the numerators and denominators.

When computing these cases on the main diagonal serially with an addition algorithm, roughly the first 40% of the fractions generated on the diagonal (until they intersect the hyperbola $n = qd = 2^p$) contain RD-hard fractions with alternating odd and even numerators. The remainder of the diagonal contains fractions with only even numerators. Figure 6 describes an algorithm for generating these fractions for any $p \geq 5$ and all k.

#define VPRINT(n, d) if (n even or $n < 2^p$) print $\frac{n}{d}$

/* INITIALIZATION */
if (p is odd) $even$ = 0; else $even$ = 1;

$n = 2^{p-1} + 2^{(p+1+even)/2} + 1 + even$;
$d_1 = 2^{p-1} + 2^{(p-1+even)/2} - 1$; $d_2 = 2^{p-1} + 2^{(p-1+even)/2} + 1$;
$n_{inc} = 2^{(p+1+even)/2} + 1 + even$;
$n_{inc\_inc} = 2 + 2 \times even$; $d_{inc} = 2^{(p-1+even)/2}$;
stop = 2^p;

/* COMPUTATION */
while (d_2 < stop) {

/* RD-hard: Main Diagonal */
VPRINT(n,d_1); VPRINT(n,d_2);

/* RD-hard: 3's Comp Symmetry: Cross Diagonal */
VPRINT($d_1 * 3 - n$,d_1); VPRINT($d_2 * 3 - n$,d_2);

/* Add RN-hard Calculation Here */

$n_{inc} = n_{inc} + n_{inc\_inc}$;
$n = n + n_{inc}$;
$d_1 = d_1 + d_{inc}$; $d_2 = d_2 + d_{inc}$;
}

Figure 6: Calculation of the Diagonal RD-hard Fractions

In [MM01], we showed a relationship between RD-hard fractions and RN-hard fractions with the same denominators. Through this relationship and RN-hard 3's complement symmetry, a similar number of RN-hard fractions can concurrently be generated via simple additions. Figure 7 shows the additional steps needed to generate the RN-hard fractions. These steps should be placed where indicated in Figure 6.

```
/* Calculate RN-hard Numerators */
if (n is odd )
  rn1 = (d1+n)/2;   rn2 = (d2+n)/2;
else
  rn1 = 2d1 - n/2;   rn2 = 2d2 - n/2;
}

/* Print RN-hard Fractions */
VPRINT( rn1,d1);    VPRINT( rn2,d2);

/* RN-hard: 3's Complement Symmetry */
VPRINT( d1 * 3 - rn1,d1); VPRINT( d2 * 3 - rn2,d2);
```

Figure 7: Calculation of RN-hard Fractions

Using the steps of Figures 6 and 7, we generated 330,917,692 RD-hard test cases and 168,902,564 RN-hard test cases for double precision p=53 in 5 minutes using a 90MHz Pentium processor.

A different set of initial conditions in Figure 6 will generate another well distributed set of test cases along different diagonals in the QD-plot. The initial conditions in Figure 8 along with the algorithms in Figures 6 and 7 generate at least $.75 \times 2^{(p+1)/2}$ RD-hard fractions and at least $.75 \times 2^{(p+1)/2}$ RN-hard fractions for odd $p \geq 7$ or even $p \geq 10$.

/* INITIALIZATION */
if (p is odd) $even$=0; else $even$ = 1;

$$n = 2^{p-1} + 2^{p-3-even} + 2^{(p-1-even)/2} + 2^{(p-7-even*3)/2};$$
$$d_1 = 2^{p-1} + 2^{(p-3-even)/2} - 1;$$
$$d_2 = 2^{p-1} + 2^{(p-3-even)/2} + 2^{p-3-even} + 1;$$
$$n_{inc} = 2^{(p+1-even)/2} + 2^{(p-5-even*3)/2}$$
$$n_{inc\_inc} = 2 - even; \quad d_{inc} = 2^{(p-1-even)/2};$$
stop = 2^p;

Figure 8: Alternate Test Case Generation

Using these initial conditions along with the algorithms of Figures 6 and 7, we generated an additional 150,994,994 RD-hard test cases and an additional 138,736,028 RN-hard test cases for double precision p=53 in 2.5 minutes using a 90MHz Pentium.

The difficulty of rounding a hard to round fraction is determined by its denominator since it has a distance of $\frac{1}{d2^{p-1}}$ from an endpoint or a distance of $\frac{1}{d2^p}$ from a midpoint. For each odd d, $2^{p-2}+1 \leq d \leq 2^p - 1$, there is one even numerator RD-hard fraction and sometimes also an odd numerator RD-hard fraction. It is well known that within the range [1,2) the hardest to round fraction for round to nearest has a denominator of $2^p - 1$ and is adjacent to $\frac{1}{1}$ in the $Q_{p \times p}$ series. In a previous paper [MM00], we showed the second hardest to round fraction for round to nearest has a denominator of $2^p - 3$ and is adjacent to $\frac{5}{3}$ in the $Q_{p \times p}$ series, while the third has a denominator of $2^p - 5$ and is adjacent to either $\frac{7}{5}$ or $\frac{9}{5}$, and so on. A similar statement can be proved for directed rounding.

Observation 13: For directed rounding, the hardest to round fraction has a denominator of $2^p - 1$ and is adjacent to $\frac{2^p-1}{2^{p-1}}$, the representable p bit endpoint which is a child of $\frac{2}{1}$. The second hardest to round fraction has a denominator of $2^p - 3$ and is adjacent to $\frac{2^p-3}{2^p}$, the p bit endpoint which is a child of $\frac{4}{3}$; and the third has a denominator of $2^p - 5$ and is adjacent to the p bit endpoint which is a child of either $\frac{6}{5}$ or $\frac{8}{5}$, etc. □

We previously outlined an algorithm for directly calculating these RN-hard fractions in order of difficulty for round to nearest[MM00]. This algorithm accepted a precision p and a rank n, and returned the even numerator and the denominator of the n^{th} hardest to round fraction. The algorithm was then placed in a loop to generate the entire set of fractions ordered by rounding difficulty. Any hard to round fraction with an odd numerator will tie in difficulty to a hard to round fraction with an even numerator. This odd numerator fraction may be calculated using the 3's complement symmetry.

A modified version of this algorithm is provided in Figure 9 which produces the n^{th} hardest fraction for directed rounding This algorithm uses the GCD algorithm to find the necessary non-closest parent and requires 7μs to calculate a 24 or 53 bit fraction on a 500 MHz Pentium III.

for i=3 to $2^{\frac{p}{2}}$, i is odd {

$\frac{n_s}{d_s}$ = non_closest_parent $(\frac{2^p-i}{2^{p-1}-i})$

if n_s is odd

$n_{DRH} = 2^p - i + n_s$

else

$n_{DRH} = (2^p - i) \times 2 - n_s$

The next hard to round fraction is $\frac{n_{DRH}}{2^p-i}$, (n_{DRH} is even; check for 3's complement)

}

Figure 9: Calculation of the n^{th} Hardest Fraction for Directed Rounding

Table 10 shows the 10 hardest to round fractions for directed rounding with precisions of 24, 53, and 64 bits.

Table 10 Hardest to Round Fractions For Standard Precisions

Rank	Directed Rounding		
	p=24	p=53	p=64
1	$\frac{33554428}{16777215}$	$\frac{18014398509481980}{9007199254740991}$	$\frac{36893488147419103228}{18446744073709551615}$
2	$\frac{22369618}{16777213}$	$\frac{12009599006321318}{9007199254740989}$	$\frac{24595658764946068818}{18446744073709551613}$
3	$\frac{26843538}{16777211}$	$\frac{10808639105689184}{9007199254740987}$	$\frac{29514790517935282578}{18446744073709551611}$
4	$\frac{28760930}{16777209}$	$\frac{12867427506772836}{9007199254740985}$	$\frac{21081993227096630410}{18446744073709551609}$
5	$\frac{29826146}{16777207}$	$\frac{14011198840708196}{9007199254740983}$	$\frac{20496382304121724008}{18446744073709551607}$
6	$\frac{27453608}{16777205}$	$\frac{11463708142397612}{9007199254740981}$	$\frac{30185581211524720808}{18446744073709551605}$
7	$\frac{30973298}{16777203}$	$\frac{15242952584946272}{9007199254740979}$	$\frac{25541645640520917604}{18446744073709551603}$
8	$\frac{31317442}{16777201}$	$\frac{9607679205057042}{9007199254740977}$	$\frac{34433922270924496322}{18446744073709551601}$
9	$\frac{31580610}{16777199}$	$\frac{9537034505019856}{9007199254740975}$	$\frac{34723282962276803010}{18446744073709551599}$
10	$\frac{19426228}{16777197}$	$\frac{16118146034799636}{9007199254740973}$	$\frac{19417625340746896418}{18446744073709551597}$

Conclusions

We have shown that the class of division cases which are most sensitive to rounding error can be enumerated for both directed rounding and round to nearest. The cost of exhaustive enumeration is one $p \times p$ bit GCD computation per case. When plotting these division test cases on a 2-dimensional quotient vs divisor graph, various symmetries and linear progressions become evident. By exploiting these symmetries and linear progressions, large representative subsets of hard to round fractions may be efficiently determined via simple additions.

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