FORMAL SYSTEMS OF NUMERALS

ARMEN GABRIELIAN
Independent Consultant
1890 Sixteenth St.
Newport Beach, Calif. 92663

Summary

A new system of numerals is introduced for representing numbers in base 2^N, for $N \leq 8$. The new notation greatly simplifies arithmetical operations on numbers. For example, for N=3(4) one obtains a notation for octal (hexadecimal) numbers in which one can perform addition and multiplication much more easily than in the standard notation. For N=8 one obtains a practical way of representing numbers to the base 256. A simplification of the decimal notation is also presented.

1. Introduction

The Indo-Arabic decimal numeral system has been the most widely accepted notation for representing numbers for many centuries. Quite different systems, however, were popular in the past and the binary, octal and hexadecimal systems have recently attained great prominence in connections with computers. There are two basic features that are characteristic of all the major numeral systems in use today. First, each of them has a relatively small fixed base—the largest being 16 for the hexadecimal. Secondly, each contains distinct and unrelated abstract symbols that represent its digits. Specifically, the binary, octal, decimal and hexadecimal systems use, respectively, the sets of symbols {0,1} , {0,1,2,3,4,5,6,7}, {0,1,2,3,4,5,6,7,8,9} , and {0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F}. In contrast, the numeral systems of the past did not always have a unique small base and the symbols for digits were often made up of more elementary symbols. The following are some examples:*

(1) The Babylonian numeral system was a combination of sexagesimal (base 60) and decimal. It used just the two elementary symbols I and < , which represented decimal 1 and 10, respectively. The main difficulty with this system was that it did not possess a symbol for zero and its numerals did not have unique interpretation.
Examples. III=3, IIII=4, IIIIIII=7, <I=11,

I<<IIIII=1x60+25=85 or
=1x60^2+25x60=5100.

(2) The hieroglyphic numeral system used by the Egyptians had special symbols for various powers of 10. For instance,

𓏺=1, 𓎆=10, 𓍢=100, 𓆼=1000, and Ω=10^7,
so that

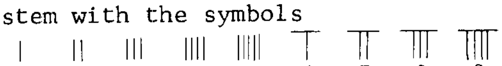
ΩΩ∩∩III=20000123.

*Historical references are listed at the end of the paper.

(3) The Attic system of the Greeks had the following symbols:
Ι=1, Γ=5, Δ=10, 𐅄=50, Η=100,
𐅅=500, Χ=1000, 𐅆=5000, Μ=10,000,
𐅇=50,000.
Thus, e.g., 𐅆ΧΧ𐅄ΔΓΙΙΙ=7068.

(4) The Roman numeral system, of course, is quite well known and is still used occasionally. It has the following symbols:
I=1, V=5, X=10, C=100, D=500, M=100
For example, MMMDLXVIII=3568.

(5) The Chinese had a veriety of numeral systems. The "stick numbers" and their Sangi derivatives in Japan (beginning about A.D.60 used an Arabic-like decimal place-value system with the symbols

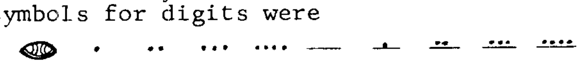


1 2 3 4 5 6 7 8 9.

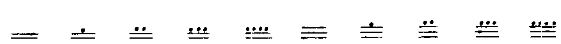
In an even position from the right, these were converted to

Therefore, e.g., 𝍯𝍧=78, and 𝍪𝍥𝍰𝍢=2683. The symbol for zero, "O", made its way to China from India in the middle of the 13-th century. The first recorded[6] history of zero in India itself is A.D. 870.

(6) The Mayans of Central America had, perhaps, the most logical and advanced numeral system of early civilization. It wa a vigesimal (base 20) place-value system superimposed on a quinary (base 5) notation. The Mayan priests had devised a symbol for zero as early as 300 or 400 B.C.. Their symbols for digits were

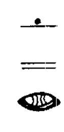


0 1 2 3 4 5 6 7 8 9



10 11 12 13 14 15 16 17 18 19.

Mayan numerals were written vertically and t only irregularity in their system was that, in recording of time, the third position represented 18 and not 20. That is,



= 6x20x18x20=43200 +
10x18x20= 3600
0x20= 0
3= 3
46803

What is remarkable about the Mayan numeral system is its elegant simplicity. The representation of 5 by — is a distinct improvement over the Chinese ǀǀǀǀǀ since it allows the principle of superposition to be used in performing of addition. For example, ≐ + ⋯ = ≡ and ═ + ···· = ⩸ . The difficulty that it shares with the other numeral systems of the past is its excessive dependence on ideographic symbolism. On the other hand, the Indo-Arabic abstract numeral systems in use today require complete memorization of addition and multiplication tables and are not practical for large bases. Even for base 16, which is the natural system for many computers, few people can perform multiplication without conversion to the decimal representation.

In Section 2 of this paper we propose a formal system of numerals in which numbers to base 2^N ($N \leq 8$) can be represented without the necessity of memorizing more than N elementary symbol, plus zero. In Section 3 we present examples of such formal notation for the quaternary, octal, hexadecimal and base 256 (N=8). In Section 4 we consider a possible formal representation of the decimal numerals. However, since 10 is not a power of two the degree of formalization of this is much less than the other systems considered in this paper. Finally, in Section 5 we discuss the advantages and disadvantages of formal numeral systems.

2. The Concept of a Formal Numeral System

Our entire system of numerals for a base B, with $B=2^N$ for $N \leq 8$, is based on the simple idea of associating an elementary stroke for each power of 2, called elementary digit, that is smaller than B. Then to form a symbol for a digit $2^{N_1}+\cdots+2^{N_k}$, where $N_1 < \cdots < N_k < N$, we simply combine the elementary strokes for $2^{N_1},\ldots,2^{N_k}$. Figure 1 defines the elementary strokes for powers of 2 considered in this paper.

ǀ	¯	⟍	⟋	⌴	δ	⊓	♀
1	2	4	8	16	32	64	128

Figure 1. Elementary strokes and digits.

The rules of combination of the strokes are illustrated by the representation of the digit 255 by ⧩ in base 256. For a smaller base, of course, the appropriate subset of strokes in Figure 1 would be used. Some examples of these are illustrated in Figure 2.

Base 4	ǀ	¯	┐												
Octal	ǀ	¯	┐	⟍	⟍ǀ	⟍¯	⟍┐								
Hex	ǀ	¯	┐	⟍	⟍ǀ	⟍¯	⟍┐	⟋	⟋ǀ	⟋¯	⟋┐	╳	╳ǀ	╳¯	╳┐
Dec.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Fig. 2. Formal digits in bases 4, 8 and 16.

Note that although our numeral system is based on the binary coding of numbers, it uses on the average far fewer elementarary symbols than in the binary representation. For instance, in base 8, ¯=10(binary), ⟍=100, ⟍ǀ=101, ⟍¯=110, ⟍┐=111 and ⟍ǀ⟍¯=101110. Thus, the number of elementary strokes in a digit is equal to the number of 1's in its binary expansion.

An alternate way of deriving the same concept of a formal numeral system is to define a set Σ of N elementary strokes whose subsets would denote the digits of the system. There are 2^N possible subsets of Σ, with the empty subset being denoted by 0. For N=2 let Σ={ǀ,¯}. Then { }, {ǀ}, {¯}, and {ǀ,¯} are the possible subsets of Σ. These can be represented, respectively, by 0, ǀ, ¯, and ┐. The most natural interpretation of a compound symbol is to assume that it represents the sum of numbers denoted by its constituent elementary strokes. Thus, we conclude that if ǀ represents the integer 1, then ¯ and ┐ represent 2 and 3, respectively. This argument can be extended to arbitrary N. For example, for N=3, if we assume that ⟍=4, then it follows that ⟍ǀ=5, ⟍¯=6 and ⟍┐=7. The only freedom is the rule for combining the elementary strokes.

3. Examples of Formal Numeral Systems

The binary formal numeral system (N=1) is identical to the standard notation except that 1 is replaced by ǀ. We now consider, in order, the specific formal numeral systems obtained when N takes on the values 2, 3, 4 and 8.

3.1. The Quaternary System (Base 4, N=2)

For base 4 the set Σ of elementary strokes is {ǀ,¯}. We define ǀ=1 (decimal), ¯=2 and ┐=3. Figure 3 then presents the addition and multiplication tables for this numeral system.

+	ǀ	¯	┐
ǀ	¯	┐	ǀ0
¯	┐	ǀ0	ǀǀ
┐	ǀ0	ǀǀ	ǀ¯

×	ǀ	¯	┐
ǀ	ǀ	¯	┐
¯	¯	ǀ0	ǀ¯
┐	┐	ǀ¯	¯ǀ

Fig. 3. Addition & multipl. tables for base 4

Since addition and multiplication are associative and distributive only operations between the elementary digits of Figure 3 need be memorized. In case of addition this can even be further reduced by noting that the addition of distinct elementary digits results simply in the combination of their associated strokes. Figure 4 then presents the reduced addition and multiplication tables for base 4.

X	ǀ	¯
X+X	¯	ǀ0

×	ǀ	¯
ǀ	ǀ	¯
¯	¯	ǀ0

Fig. 4. Reduced add. & mult. tables(base 4)

To obtain the sum of two non-elementary digits we first add their ı strokes and then the ‾ strokes. This is similar to the addition process with abstract symbols which proceeds from the right-most digit to the left. The important point to remember is that if two digits have no strokes in common, then their sum is denoted by the combination of all their strokes. For the quaternal system this situation arises only for ‾+ı=⌉ and addition with zero. However, the range of application of this rule increases considerably as the base becomes larger.

Multiplication by a non-elementary digit can be performed in terms of its elementary components as in the usual multiplication of decimal numbers. The only difference is that subproducts must be aligned on the right. For example,

```
  ⌉ı‾ x        312 x
    ⌉            3
  ⌉ı‾          312    (=312x1)
 ı‾⌉0         1230    (312x2)
 ‾‾0‾         2202
```

To multiply by a number containing two or more digits we simply use this scheme along with the standard rule of multiplication.

Example.

```
  ⌉ı‾ x        312 x
   ⌉‾           32
 ı‾⌉0         1230    (312x2)
  ⌉ı‾          312    (312x1)
 ı‾⌉0         1230    (312x2)
‾⌉⌉ı0        23310
```

Besides the fact that the complete addition and multiplication tables need not be memorized, an important advantage of formal numerals is that a digit can be constructed piecemeal. Consider the addition ‾ + ‾ + ⌉ . One can immediately conclude that the right-most digit of the result contains the elementary stroke ı . Then by noting that ‾+‾+‾=ı‾ , the final result ı⌉ can be obtained. This piecemeal construction of digits is even more useful in the handling of carry figures in multiplication. Since a carry figure must be smaller than a multiplier digit or any multiple thereof, we can formulate the following rule for it.

Carry Figure Rule for Formal Numerals

Let a and b be formal digits and let e be an elementary digit. Then if $exb=d_2d_1$ and $exab=d_3d_2'd_1$, d_1, d_2, d_2' and d_3 being digits, then d_2' contains d_2. (Actually, d_2 must be a single elementary stroke.)

In other words, the carry figure can be directly written in the result. In contrast, when the standard abstract symbols for numerals are used, the carry figure d_2 must be recorded separately and then added to the product exa.

Examples.

```
(1)    ı‾ x
        ‾
       ⌉0
```

Here, the stroke ı in ⌉ can be recorded as the carry figure from the product ‾x‾ , irrespective of the product ı x ‾ .

```
(2)    ı⌉‾ x
         ‾
       ⌉⌉0
```

Here, the same phenomenon occurs twice.

3.2. The Octal System (N=3)

In the octal case, we extend the set Σ of elementary strokes of the quaternary system by introducing the stroke \ which denotes the number 4. The rules of combination are defined by the representation of 7 by ∇ . The octal digits then are

ı	‾	⌉	\	∨	ᐸ	∇
1	2	3	4	5	6	7.

As in the quaternary case, we can restrict our attention to elementary digits in defining the addition and multiplication tables (Fig.5). For addition, the trivial cases of the addition of distinct elementary digits are again not included.

X	ı	‾	\
X+X	‾	\	ı0

x	ı	‾	\
ı	ı	‾	\
‾	‾	\	ı0
\	\	ı0	‾0

Fig. 5. Addition & mult. tables for base 8.

Examples of addition.

(1) ı + \ = ∨ , ‾ + \ = ᐸ , ‾ + ı = ⌉

(2) ᐸ + ı = ∇ , ∨ + ‾ = ∇ , ⌉ + \ = ∇

(3) ∨ + ı = ᐸ , ∇ + ı = ı0 , ∨ + ∨ = ı‾

(4) \ + ᐸ + ∨ + ı0 = ‾∇

In the last example, as in the quaternary system, one can immediately conclude that ⌉ is part of the right-most digit of the resultant sum. The stroke \ can be added later to obtain ∇.

Examples of multiplication.

```
(1)    ∇ x
       ‾
      ıᐸ
```

To obtain this product, we first form ‾ x ı = ‾ and then since ‾x‾ = \ we add \ to ‾ to obtain ᐸ . Finally, ‾x\=ı0 completes the result.

```
(2)   ∨⌉∇ x          537 x
        ᐸ              6
     ı‾∇ᐸ           1276     (537x2)
     ‾∨∇\           2574     (537x4)
     \0∇‾           4072
```

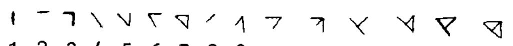
Here, the carry figure rule of Section 3.1 is applied several times. For instance, in multiplying ⌉∇ by ‾ the following steps are traversed:

```
                       ⌉∇ x
                        ‾
Step 1: ‾xı=‾           ‾
Step 2: ‾x‾=\           ᐸ
Step 3: ‾x\=ı0         ıᐸ
Step 4: ‾x⌉=ᐸ         [∇ᐸ]       Final result.
```

On the other hand, in performing addition the carry figure cannot be immediately written without regard to the digits in the next column to the left. For example, |◁ + | = ¯0, whereas, ◁ + | = |0 which has a carry figure of |.

3.3. The Hexadecimal System (N=4)

We define the set Σ of elementary strokes for hex as {|, ¯, \, /}, with / representing decimal 8. The digits then are



1 2 3 4 5 6 7 8 9 10 11 12 13 14 15.

Of course, 16 would be represented by |0 and, e.g., 33 by ¯|. Figure 6 contains the hex addition and multiplication tables.

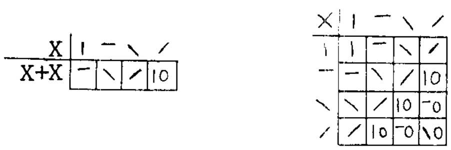
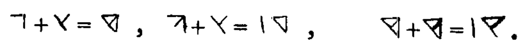


Fig. 6. Addition & multipl. tables for hex.

Examples of hex addition.



Before we present some examples of hex subtraction, we introduce a definition, and a rule that simplifies the subtraction process when a subtrahend digit is larger than its corresponding minuend digit.

Definition. Let X be a digit in a formal numeral system . Then, the complement of X, denoted by X^c, is the combination of all the elementary strokes of the system that are not in X.

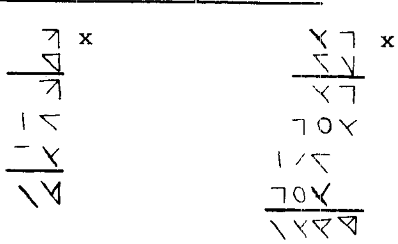
Rule of Subtraction from |0. Let X be a formal digit, then

$$|0 - X = X^c + |.$$

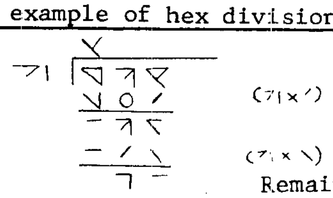
Examples of hex subtraction.

The rule of subtraction from |0 is used in the second example to obtain |0 − ⋎ = ⅂ + | = \ . Then, ◁ + \ gives ⅂\ .

Examples of hex multiplication.



An example of hex division.



3.4. A Numeral System for Base 256 (N=8)

To form a numeral system for base 256 we let Σ be the full set of strokes in Figure 1. That is, we extend the formal hex numerals by defining ⌴=16, ᵟ=32, ⌐=64 and ♀=128. The rules of combination of the strokes are defined by [symbol] which denotes 255. Thus,

=16+8+4+2=30,

=64+32+4+1=101 and

=128+4+2+1=135.

In base 256 any 8-bit number (byte) is represented by a single digit. Also, every two-digit hex number corresponds to a single digit in base 256. For instance,

$$10011101_{(2)} = [\text{symbol}]_{(16)} = [\text{symbol}]_{(256)}.$$

As in previous formal numeral systems with base a power of 2, addition and multiplication in base 256 can be performed in terms of the addition and multiplication tables for elementary digits (Figure 9).

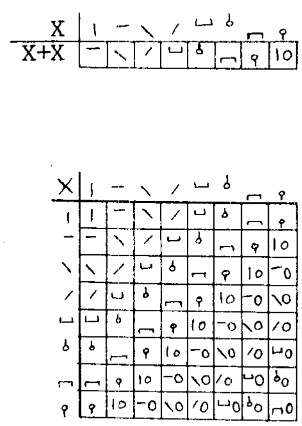
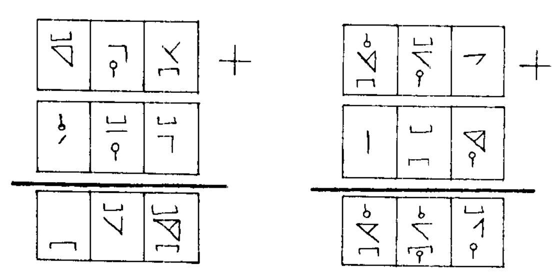
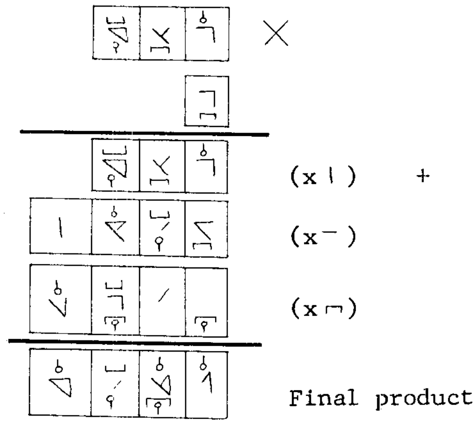


Fig. 9. Addition & multipl. tables for base 256.

Examples of addition in base 256.



An example of multiplication in base 256.



4. A Simplified Notation for Decimal Numerals

Since 10 is not a power of 2 a simple numeral system like those of Section 3 cannot be devised for decimal numbers. If the first ten numerals (including zero) of hex are taken as the decimal digits, then addition and multiplication cannot be defined in terms of these operations just on the elementary digits. Further rules would be needed and the resultant numeral system would not be significantly better than the standard abstract symbolism for decimal numbers. Using the quinary (base 5) and a variation of the binary systems, however, we will now present a possible formal numeral system for decimal numbers. This is not meant to explain the evolutionary development of the decimal system. It is intended merely as a possible notational simplification of the representation of decimal numbers.

Consider the standard quinary system with digits 1, 2, 3, 4 and the associated addition and multiplication tables as presented in Figure 7.

+	1	2	3	4
1	2	3	4	10
2	3	4	10	11
3	4	10	11	12
4	10	11	12	13

x	1	2	3	4
1	1	2	3	4
2	2	4	11	13
3	3	11	14	22
4	4	13	22	31

Fig. 7. Addition & mult. tables for base 5.

Let ⌣ denote the integer 5 and consider the new set of decimal digits

1 2 3 4 ⌣ 1̮ 2̮ 3̮ 4̮
1 2 3 4 5 6 7 8 9.

The resultant decimal numeral system will be called biquinary*. As in previous numeral systems, we can use superposition to perform biquinary addition and multiplication. We first present a conversion rule that will be used on the entries of the tables in Fig. 7.

Quinary to Biquinary Conversion Rule for Two-Digit Numbers.

Let X and Y be quinary digits. Then the quinary number XY is equivalent to the biquinary [Y/2]✱, where

(1) [Y/2] is the largest integer in Y/2.

(2) ✱ = X if Y is even.
    = ✱̮ if Y is odd.

For instance, $44_{(5)}=24_{(10)}$ and $32_{(5)}=1\breve{2}_{(10)}$.

Biquinary addition can now be performed easily by using quinary addition and the equation ⌣ + ⌣ = 10.

Examples of biquinary addition.

(1) 3+4=12 in quinary. Therefore, in the biquinary representation of decimal we have

3+4=2̮, 3+4̮=12, 3̮+4=12, 3̮+4̮=12̮.

(2) 4+4=13 in quinary. Then, in biquinary we have

4+4=3̮, 4+4̮=13, 4̮+4̮=13̮.

To perform multiplication we only need add the multiplication table of Figure 8 for ⌣ to the quinary table of Figure 7 and use superposition.

x	1	2	3	4	⌣	1̮	2̮	3̮	4̮
⌣	⌣	10	1⌣	20	2⌣	30	3⌣	40	4⌣

Fig. 8. Biquinary multiplication table for ⌣.

Examples of biquinary multiplication.

(1) 3x4=22 in quinary. Therefore, in biquinary we have

3x4=12, 3x4̮=22̮, 3̮x4=32, 3̮x4̮=22.

(2) 4x4=31 in quinary. In biquinary we have

4x4=11̮, 4̮x4=31, 4x4̮=31̮.

5. Advantages and Disadvantages of Formal Numeral Systems

The main advantages of the formal numeral systems proposed in this paper are that (1) they provide a logical relationship between numerals and their associated numbers, (2) they simplify arithmetical operation, and (3) fewer digits are needed to represent numbers when a large base is used.

We shall now discuss these three points in more detail.

(1) While most ancient numeral systems were cumbersome because of their ideographic excesses, the abstract Indo-Arabic numerals

* The standard biquinary representation of decimal digits is (01,00001), (01,00010),..., (01,10000), (10,00001), (10,00010),..., (10,10000).

are totally devoid of representational value. In a small base like 8 or 16 the formal numerals are no more complicated than the abstract numerals yet they present a clear picture of the structure of numbers. For a large base like 256 one cannot even conceive of a practical abstract numeral system.

(2) In view of the examples and the discussion in Section 3 it is clear that it is much easier to perform arithmetical operations on formal numerals than on abstract numerals. Much smaller addition and multiplication tables need be memorized and the carry figure in multiplication can be directly recorded in a product (by the rule of Section 3.1). As another example, consider divisibility by 3, 5, or 15 in hex. Since 16=15+1, a number is divisible by3, 5, or 15 if the sum of the digits in its hex representation is divisible by 3, 5, or 15, respectively. This can be ascertained much more easily if the formal hex numerals of Section 3.3 are used. For example, one can immediately determine that ᐊ1ᐸ is divisible by 15 (ᐊ). On the other hand, in its standard representation, 79E, this fact seems much less obvious.

(3) Under a uniform distribution, the average number of digits necessary to represent numbers in base 256 is close to half the number of digits required in hexadecimal* and even less compared to the decimal. Aside from this, it is convenient to have a one-digit representation of a byte which is a common unit of information in many computers. Finally, note that any digit in base 256 can easily be recorded on a single column of a computer card (a maximum of 8 perforations).

The main disadvantage of our formal numeral systems is that the existing abstract numeral systems are so well-entrenched that there will be a tremendous resistance to the introduction of a new symbolism. The Duodecimal Society of America has for many years religiously advocated the adoption of a numeral system in base 12, with T (deck) representing 10 and E (el) representing 11, with very little success. On the other hand, historically, there have been instances where formalizations that are based on logically sound foundations and are of clear utilitarian value have been widely adopted. A case in point is the metric system. It is therefore possible that a system of numerals such as the one proposed here could find some acceptance in the future.

* Only for numbers less than 16 there will be an equal number of digits.

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