A UNIFIED APPROACH TO A CLASS OF NUMBER SYSTEMS

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Abstract

A unified approach to a broad class of number systems is proposed in this paper. This class contains all positive and negative radix systems and other well-known number systems. The proposed approach enables us to develop a single set of algorithms for arithmetic operations and conversion methods between number systems.

Index Terms

Number systems, positive radix, negative radix, addition, subtraction, number conversion.

I. Introduction

Several number systems have received a great amount of attention in recent years. The conventional unsigned binary system and the 2's complement system are well-established number systems for which numerous algorithms for arithmetic operations have been developed and implemented in digital computers. Recently, the interest in negative radix number systems has increased, and different algorithms for arithmetic operations in these number systems were developed [1-5]. The idea of using negative radixes was extended lately to binary coding of decimal numbers [6].

A unified approach to all these number systems is proposed in this paper . It is shown that the above mentioned systems and some additional number systems are members of a broad class of number systems in which a mixture of positive and negative radixes is used. We first define this class of number systems and state its basic properties. Next, algorithms for addition and subtraction are developed and conversion methods between any two number systems in this class are presented.

II. Definitions and basic properties

Consider a class of n-digit, fixed base, weighted, linear number systems in which every number system is characterized by the base β which is a positive integer and by a vector Λ of length n, $\Lambda = (\lambda_{n-1}, \lambda_{n-2}, \ldots, \lambda_0)$ where $\lambda_i \varepsilon \{1,-1\}$. In such a number system characterized by the pair $<\beta,\Lambda>$ the algebraic value X of an n-tuple $(x_{n-1}, x_{n-2}, \ldots, x_0)$ where $x_i \varepsilon \{0,1,\ldots,\beta-1\}$ is defined by

$$X = \sum_{i=0}^{n-1} \lambda_i x_i \beta^i \tag{1}$$

For a given base β we have in this class 2^n distinct number systems. Among them are the positive radix number systems, i.e., $\lambda_i = +1$ for every i, and the negative radix number systems, i.e., $\lambda_i = (-1)^i$ for every i. For example, the conventional unsigned binary system is characterized by $<\beta=2,\Lambda=(1,1,\ldots,1)>$. The negabinary system is characterized by $<\beta=2,\Lambda=(\ldots,-1,+1,-1,+1)>$. Additional known number systems are included in this class. The 2's complement system is characterized by $\beta=2$ and $\Lambda=(-1,1,1,\ldots,1)$. The (8,-4,-2,1) binary coding of decimal numbers [6] is characterized by $\beta=2$ and $\Lambda=(\ldots,1,-1,-1,1,1,-1,-1,1)$.

All the number systems in this class are clearly non-redundant and complete [7], i.e., in a given number system defined by $<\beta,\Lambda>$ we have a unique representation for any number X within the range of the system. (Consequently, this class is different from the class of signed-digit number systems [8] which are redundant). The largest representable integer in the system $<\beta,\Lambda>$ is the positive number P whose representation is $(p_{n-1}, p_{n-2}, \ldots, p_0)$ where

$$p_i = \begin{cases} \beta-1 & \text{if } \lambda_i = +1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Its value is given by

$$\begin{aligned} P &= \frac{1}{2}\sum_{i=0}^{n-1}(\lambda_i+1)(\beta-1)\beta^i \\ &= \frac{1}{2}\left[\sum_{i=0}^{n-1}\lambda_i(\beta-1)\beta^i + \sum_{i=0}^{n-1}(\beta-1)\beta^i\right] \\ &= \frac{1}{2}\left[R + (\beta^n-1)\right] \end{aligned} \tag{3}$$

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where R is the algebraic value of the number $(\beta-1,\beta-1,\ldots,\beta-1)$ in the system $\langle\beta,\Lambda\rangle$. Similarly, the smallest integer is the negative number N whose representation is

$$n_i = \begin{cases} \beta-1 & \text{if } \lambda_i = -1 \\ 0 & \text{otherwise} \end{cases}$$

and the corresponding value is

$$N = \frac{1}{2}\sum_{i=0}^{n-1} (\lambda_i - 1)(\beta-1)\beta^i$$

$$= \frac{1}{2}\left[R - (\beta^n - 1)\right] \tag{4}$$

The number of integers in the range $N \leq X \leq P$ is $P-N+1=\beta^n$ and the range of the system $\langle\beta,\Lambda\rangle$ is in general asymmetrical. A measure of the asymmetry can be the difference $P-|N|$ which equals

$$P + N = R = \sum_{i=0}^{n-1} (\beta-1)\lambda_i\beta^i$$

Example: The negabinary system $\langle\beta=2,\Lambda=(\ldots,-1,+1,-1,+1)\rangle$ is asymmetrical and for n even there are twice as many negative numbers as positive ones [3,4]. If we prefer to have more positive numbers we can use instead the system $\langle\beta=2,\Lambda=(+1,-1,\ldots,+1,-1)\rangle$. Two binary systems are nearly symmetrical. One is the 2's complement system $\langle\beta=2,\Lambda=(-1,1,\ldots,1)\rangle$ for which we have $P-|N|=R=-1$. The other is the system $\langle\beta=2,\ \Lambda=(+1,-1,\ldots,-1)\rangle$ for which we have $P-|N|=R=+1$.

A complement of a number in the system $\langle\beta,\Lambda\rangle$ can be defined as follows. Let $\bar{x}_i \triangleq (\beta-1)-x_i$; the complement $\bar{X}$ of the number $X=\sum_{i=0}^{n-1} x_i\lambda_i\beta^i$ is

$$\bar{X} = \sum_{i=0}^{n-1} \bar{x}_i\lambda_i\beta^i$$

$$= \sum_{i=0}^{n-1} (\beta-1)\lambda_i\beta^i - X$$

$$= R - X \tag{5}$$

For example, in the 2's complement method $R=-1$ and $\bar{X}=-1-X$ or $-X=\bar{X}+1$. The complement $\bar{X}$ can be used in order to perform subtraction by addition in the following way

$$\overline{(\bar{X}+Y)} = R - (\bar{X}+Y)$$

$$= R - (R - X + Y) = X - Y \tag{6}$$

However, a more efficient algorithm for subtraction is presented in the next section.

III. Addition and subtraction

Different algorithms for addition and subtraction in positive radix and negative radix number systems have been developed [1,2,4]. We show that a unified treatment of all number systems in the previously defined class is possible and that a single addition-subtraction algorithm can be developed.

Let X and Y be two numbers in the system $\langle\beta,\Lambda\rangle$. We want to form the sum $S=X+Y$ and the difference $D=X-Y$. Since a mixture of positive and negative digit weights is used in our number system, we have to use borrows in addition and carries in subtraction. Hence, there is no need to distinguish between addition and subtraction and a single set of rules is developed for $S=X\pm Y$. These rules specify the values of the sum digit s_i, the carry c_{i+1} and the borrow b_{i+1} from the values of x_i, y_i, the carry in c_i and the borrow in b_i. The basic equation that should be satisfied is

$$\lambda_i(x_i \pm y_i) + c_i - b_i = \lambda_i s_i + \beta c_{i+1} - \beta b_{i+1} \tag{7}$$

Let $z_i \triangleq \lambda_i(x_i \pm y_i) + c_i - b_i$, then the rule for obtaining s_i is

$$s_i = (\lambda_i z_i) \bmod \beta \tag{8}$$

For c_{i+1} and b_{i+1} we have two cases.

Case 1: $z_i \geq \lambda_i s_i$

$$\text{then } b_{i+1}=0 \text{ and } c_{i+1}=(z_i-\lambda_i s_i)/\beta \tag{9}$$

Case 2: $z_i < \lambda_i s_i$

$$\text{then } c_{i+1}=0 \text{ and } b_{i+1}=(\lambda_i s_i - z_i)/\beta \tag{10}$$

Clearly the carry and borrow outputs of stage n-1 indicate overflow and underflow, respectively.

Example: Let X=1614097 and Y=0416034 and be two numbers in the negadecimal system to be added (this example is taken from [1]). We obtain

+1	-1	+1	-1	+1	-1	+1	λ_i
1	0	1	1	1	0	0	b_i
0	0	0	0	0	1	0	c_i
1	6	1	4	0	9	7	x_i
0	4	1	6	0	3	4	y_i
0	-10	1	-11	-1	-11	11	z_i
0	0	1	1	9	1	1	s_i

For the binary number systems $\langle\beta=2,\Lambda\rangle$ a truth table can be constructed from equations (8)-(10) and logical expressions can be derived. For convenience, we replace the vector Λ by a binary vector whose elements satisfy

$$\gamma_i = \frac{1}{2}(1+\lambda_i);\ i = 0, 1, \ldots, n-1 \tag{11}$$

Thus, $\gamma_i=0$ if $\lambda_i=-1$ and $\gamma_i=1$ if λ_i+1. We define another binary variable a which satisfies a=0 for an add operation and a=1 for subtraction. Using this notation the resulting truth table is shown in Table I and the logical expressions are

$$s_i = x_i \oplus y_i \oplus (c_i+b_i) \tag{12}$$

$$c_{i+1} = \delta_i(u_i+y_i)c_i + \bar{\delta}_i\bar{u}_i\bar{y}_i c_i + \delta_i u_i y_i \bar{b}_i \tag{13}$$

$$b_{i+1} = \bar{\delta}_i(u_i+y_i)b_i + \delta_i\bar{u}_i\bar{y}_i b_i + \bar{\delta}_i u_i y_i \bar{c}_i \tag{14}$$

where $u_i=x_i \oplus a$ and $\delta_i=\gamma_i \oplus a$.

Table I

b_i	c_i	x_i	y_i	s_i	$c_{i+1}b_{i+1}$ ①	②	③	④
0	0	0	0	0	0 0	0 0	0 0	0 0
0	0	0	1	1	0 0	0 0	1 0	0 1
0	0	1	0	1	0 0	0 0	0 0	0 0
0	0	1	1	0	0 1	1 0	0 0	0 0
0	1	0	0	1	1 0	0 0	1 0	0 0
0	1	0	1	0	0 0	1 0	1 0	0 0
0	1	1	0	0	0 0	1 0	0 0	1 0
0	1	1	1	1	0 0	1 0	1 0	0 0
1	0	0	0	1	0 0	0 1	0 0	0 1
1	0	0	1	0	0 1	0 0	0 0	0 1
1	0	1	0	0	0 1	0 0	0 1	0 0
1	0	1	1	1	0 1	0 0	0 0	0 1

① $\gamma_i = 0$ ($\lambda_i=-1$) in addition (a = 0)

② $\gamma_i = 1$ ($\lambda_i=+1$) in addition (a = 0)

③ $\lambda_i = 0$ ($\lambda_i=-1$) in subtraction (a = 1)

④ $\gamma_i = 1$ ($\lambda_i=+1$) in subtraction (a = 1)

The general equations (12)-(14) can be reduced to any given binary system $\langle\beta=2,\Lambda\rangle$. For example, in the case of the conventional binary system, i.e., $\Lambda=(1,1,\ldots,1)$, we have $\gamma_i=1$ and $\delta_i=\bar{a}$; $i=0,1,\ldots,n-1$. If we write separate logical equations for addition (a=0) we obtain $u_i=x_i$ and $\delta_i=1$, consequently

$$c_{i+1} = (x_i+y_i)c_i+x_iy_i$$

$$b_{i+1} = 0$$

$$s_i = x_i \oplus y_i \oplus c_i$$

These are the well-known equations for a Full Adder. Similarly, for subtraction (a=1) we obtain the well-known equations for a Full Subtractor, i.e.,

$$c_{i+1} = 0$$

$$b_{i+1} = (\bar{x}_i + y_i)b_i + \bar{x}_iy_i$$

$$s_i = x_i \oplus y_i \oplus b_i$$

III. Conversion methods

The known conversion methods between positive and negative bases [3,4,5] can be extended to any two number systems with the same base β but different characterizing vectors $\Lambda=\{\lambda_i\}$ and $M=\{\mu_i\}$. The extension of the serial method presented by Zohar [3] is straightforward and is therefore omitted. We extend now the parallel method introduced by Yuen [4,5].

Let X be a number whose representation $(x_{n-1}, x_{n-2},\ldots,x_0)$ in the number system $\langle\beta,\Lambda\rangle$ is given. We want to find the representation $(y_{n-1},y_{n-2},\ldots,y_0)$ of X in the number system $\langle\beta,M\rangle$ so that

$$X = \sum_{i=0}^{n-1} \lambda_i x_i \beta^i = \sum_{i=0}^{n-1} \mu_i y_i \beta^i \tag{15}$$

We rewrite (15) in the following way

$$X = \sum_{i=0}^{n-1} \lambda_i x_i \beta^i = \sum_{\substack{i \\ \mu_i=\lambda_i}} \mu_i y_i \beta^i + \sum_{\substack{i \\ \mu_i\neq\lambda_i}} \mu_i y_i \beta^i \tag{16}$$

The first term in (16) contains all the elements for which $\mu_i=\lambda_i$ and hence we may replace μ_i by λ_i. The second term contains all the elements for which $\mu_i\neq\lambda_i$ or $\mu_i=-\lambda_i$. Consequently,

$$X = \sum_{\substack{i \\ \mu_i=\lambda_i}} \lambda_i y_i \beta^i - \sum_{\substack{i \\ \mu_i\neq\lambda_i}} \lambda_i y_i \beta^i \tag{17}$$

Adding $\sum_{\substack{i \\ \mu_i \neq \lambda_i}} \lambda_i \bar{y}_i \beta^i$ to both terms in (17) yields

$$X = \left(\sum_{\substack{i \\ \mu_i = \lambda_i}} \lambda_i y_i \beta^i + \sum_{\substack{i \\ \mu_i \neq \lambda_i}} \lambda_i \bar{y}_i \beta^i \right) - \sum_{\substack{i \\ \mu_i \neq \lambda_i}} \lambda_i (\beta-1) \beta^i \tag{18}$$

The term inside the parenthesis is represented by a single n-tuple whose i-th element is y_i if $\mu_i = \lambda_i$ and is $\bar{y}_i$ if $\mu_i \neq \lambda_i$. The second term in (18) is represented by an invariant n-tuple whose i-th element is $(\beta-1)$ if $\mu_i \neq \lambda_i$ and is 0 if $\mu_i = \lambda_i$. Substituting (15) and rearranging (18) we obtain

$$\left(\sum_{\substack{i \\ \mu_i = \lambda_i}} \lambda_i y_i \beta^i + \sum_{\substack{i \\ \mu_i \neq \lambda_i}} \lambda_i \bar{y}_i \beta^i \right) = \sum_{i=0}^{n-1} \lambda_i x_i \beta^i + \sum_{\substack{i \\ \mu_i \neq \lambda_i}} \lambda_i (\beta-1) \beta^i \tag{19}$$

The conversion algorithm is now summarized in the following steps:

Step 1: Add to the given number $(x_{n-1}, x_{n-2}, \dots, x_0)$ the constant digit pattern $\sum_{\substack{i \\ \mu_i \neq \lambda_i}} \lambda_i (\beta-1) \beta^i$.

The addition is performed in the $\langle \beta, \Lambda \rangle$ system.

Step 2: For each i complement the i-th element in the result of step 1 if $\mu_i \neq \lambda_i$.

Step 1 of the algorithm is illustrated in the following example.

$$\begin{array}{ccccc}
\Lambda & +1 & -1 & +1 & -1 \\
M & +1 & -1 & -1 & +1 \\
 & x_3 & x_2 & x_1 & x_0 \\
+ & 0 & 0 & (\beta-1) & (\beta-1) \\
\hline
 & y_3 & y_2 & \bar{y}_1 & \bar{y}_0
\end{array}$$

In step 2 we complement all $\bar{y}_i$ elements. The two steps of the algorithm are greatly simplified for binary systems. In this case the constant bit pattern is $\Lambda \oplus M$ which is defined as the Exclusive OR of the binary equivalents of Λ and M. Step 2 is simply an Exclusive OR operation between the result of step 1 and the constant bit pattern.

Example: Consider the conversion of the number 1111 in the 2's complement method $\langle \beta=2, \Lambda=(-1,+1,+1,+1) \rangle$ to negabinary $\langle \beta=2, M=(-1,+1,-1,+1) \rangle$. The constant bit pattern is $\Lambda \oplus M=(0,0,1,0)$ and the two steps give, successively, 0001 and 0011. The algebraic value of the number converted is -1.

IV. Conclusions

A unified approach to a class of number systems that contains the well-known and widely-used number systems has been presented. Such an approach allows a unified treatment of arithmetic operations in various number systems and thus enables the design of a single arithmetic unit capable of performing operations in several number systems. Additional arithmetic operations, such as multiplication and divisions are currently under investigation.

V. References

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