BASIC DIGIT SETS FOR RADIX REPRESENTATION OF THE INTEGERS*

David W. Matula

Department of Computer Science
Southern Methodist University
Dallas, Texas 75275

Abstract

Let Z denote the set of integers. A digit set $D \subset Z$ is basic for base $\beta \in Z$ if the set of polynomials $\{d_m\beta^m + d_{m-1}\beta^{m-1}+\ldots+d_1\beta+d_0 \mid d_i \in D\}$ contains a unique representation for every $n \in Z$. We give necessary and sufficient conditions for D to be basic for β. We exhibit efficient procedures for verifying that D is basic for β, and for computing the representation of any $n \in Z$ when a representation exists. There exist D,β with D basic for β where $\max\{|d| \mid d \in D\} > |\beta|$, and more generally, an infinite class of basic digit sets is shown to exist for every base β with $|\beta| \geq 3$. The natural extension to infinite precision radix representation using basic digit sets is considered and a summary of results is presented.

Key Words and Phrases: Base, Digit Set, Radix Representation, Non-Standard Number Systems, Finite and Infinite Precision.
CR CATEGORIES 5.30, 6.1, 5.11, 3.15

I. Introduction

The development of positional number systems has a rich history. Knuth [2, pp. 162-180] presents a recent survey noting significant contributions from established and amateur mathematicians. Although bases such as 60 and 12 were used in antiquity, most of the alternatives to standard decimal representation are of rather recent vintage. Knuth attributes to Pascal (ca.1600) the fact that any positive number could serve as radix. Positional number systems with negative digits were introduced in the early 1800s and the architecturally interesting pure balanced ternary system first appeard in an article of Lalanne in 1840[2,3].

The use of a negative base did not appear until the 1950s when several authors independently introduced the concept [2, p. 171]. Complement representation also became much discussed in this period as an alternative to sign magnitude in consideration of arithmetic computer architecture. The arithmetic of numbers represented in positional notation has a firm foundation derived from the theory of polynomial arithmetic that readily allows these extensions to negative bases and/or negative digit values, complement representation, and digit values in excess of the base[4]. Our primary concern in this paper is the characterization and computation of those integral valued base and digit set pairs that provide complete and unique finite radix representation of the integers.

In section II we introduce the integer radix representation system $P_I[\beta,D]$ as the set of radix polynomials in the integer valued base β with coefficients from the finite set of allowed integer digit values D, where $0 \in D$. Thus $P \in P_I[\beta,D]$ implies $P = d_m[\beta]^m+d_{m-1}[b]^{m-1}+\ldots+d_0$, where $d_i \in D$ for $0 \leq i \leq m$. It is stressed that $P_I[\beta,D]$ is a set of polynomial expressions, not real numbers, to afford a proper treatment of redundant representation. The digit set D is then defined to be basic for base β if the members of $P_I[\beta,D]$ are, through evaluation, in one-to-one correspondence with the integers. For D to be a basic digit set for base β we first show the necessity that D be a complete residue system modulo $|\beta|$, and secondly the necessity that D contain no non-zero multiples of $\beta-1$. When D is basic for base β, it is noted that the n-digit base β numbers with digits from D then evaluate to a set of integers that must constitute a basic digit set for base β^n, hence, by the former statement, be free of non-zero multiples of β^n-1. Our major result is then the sufficiency of the above conditions stated as a fundamental characterization theorem: D is a basic digit set for base β if and only if D is a complete residue system modulo $|\beta|$ with $0 \in D$ where the n-digit base β numbers with digits from D contain no non-zero multiples of β^n-1 for any $n \geq 1$.

For the base β and digit set D which is a complete residue system modulo $|\beta|$, we discuss in section III a simple computational procedure for determining the radix polynomial $P \in P_I[\beta,D]$ of value i when such a P exists. Furthermore, we show that the degree of such a P can grow at most logarithmically with i and linearly with the ratio of the maximum digit magnitude to the base. A simple computational procedure to confirm whether or not a given digit set D is basic for base β relying on the computation of representations for a small set of integer values is presented, yielding the re-

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sult that the determination of whether or not the digit set D is basic for base β can be accomplished in $O(|\beta| + \max\{|d| \mid d \in D\}/|\beta|)$. The digital digraph is introduced to illustrate the computational procedures of radix representation determination and basic digit set confirmation.

Specific classes of basic digit sets are described in IV. For the base β, if the digit set D has no digit value with magnitude exceeding $|\beta|-1$, then D is shown to be basic iff D is a complete residue system with -1 and 1 in D for $\beta = |D|$, or with -1 or 1 in D for $\beta = -|D|$. Thus there are $2^{|\beta|-3}$ such basic digit sets for $\beta \geq 3$, and $3 \times 2^{|\beta|-3}$ such basic digit sets for $\beta \leq -3$. Results of de Bruijn[1] on alternating digit binary representation effectively established the existence of an infinite class of basic digit sets for base 4 when the maximum digit magnitude is allowed to be larger than the base. We describe an infinite class of basic digit sets for every positive and negative base β for $|\beta| \geq 3$.

Then in section V we consider the infinite precision radix representation system $P_\infty[\beta,D]$ for bases β and digit sets D where the radix polynomial P is in $P_\infty[\beta,D]$ if and only if

$$P = d_m[\beta]^m+d_{m-1}[\beta]^{m-1}+\ldots+d_1[\beta]+d_0+d_{-1}[\beta]^{-1}+\ldots,$$

$$d_i \in D \text{ for } i \leq m.$$

The details of radix representation of the reals are beyond the scope of this paper, but several results are summarized to indicate the type of results obtainable. In particular for the digit set D which is basic for base β, let S be the set of real numbers with redundant infinite precision representations. Then:

(i) S is at least countable and can be uncountable,

(ii) $x \in S$ can have more than two but at most a finite number of representations in $P_\infty[\beta,D]$,

(iii) S can contain no $|\beta|$-ary number, i.e. no number x of the form $x = i\beta^j$ for any intergers i,j.

II. Radix Representation of the Integers

For a given integral base β, we seek those sets D of integral valued digits for which standard base β radix representation using digits from D provides a unique representation for every integer. A brief review of radix polynomial terminology is helpful[4].

Let Z be the integers. A polynomial over Z in the indeterminant x is a formal expression

$$P(x)=a_m x^m+a_{m-1}x^{m-1}+\ldots+a_1x+a_0, \quad a_i \in Z \text{ for } 0 \leq i \leq m, \tag{1}$$

where either (i) $a_m \neq 0$ and m is the degree of P(x), or (ii) $a_i = 0$ for all $i \geq 0$ and $P(x) = 0$ is the zero polynomial which is taken to have degree negative infinity. For radix representation, a base β is a positive or negative integer with $|\beta| \geq 2$, and a digit set $D \subset Z$ is a finite set of integers with $0 \in D$. A base β integer radix polynomial over D is then either the zero polynomial or a polynomial in β over D of degree $m \geq 0$, i.e.

$$P([\beta])=d_m[\beta]^m+d_{m-1}[\beta]^{n-1}+\ldots+d_1[\beta]+d_0, \quad d_i \in D \text{ for } 0 \leq i \leq m,\ d_m \neq 0. \tag{2}$$

Notationally, the brackets are maintained about the base in (2) to stress that the radix polynomial is a formal expression even though the value of the base may be expressly substituted. Hence, notationally,

$4\times[10]^2+5\times[10]+7\neq3\times[10]^2+15\times[10]+7$ denotes non-identical radix polynomials,

$4\times10^2+5\times10+7=3\times10^2+15\times10+7$ denotes equal real values.

The integer radix representation system $P_I[\beta,D]$ is the set of all base β integer radix polynomials over the digit set D.

Thus, for example, $4 \times [10]^2 + 5 \times [10] + 9$ is a base 10 integer radix polynomial and is a member of $P_I[10,\{0,1,2,3,4,5,6,7,8,9\}]$, the standard decimal integer radix representation system. The base 3 integer radix polynomial $[3]^4-[3]^3-[3]^2+[3]+1$ is a member of the balanced ternary integer radix representation system $P_I[3,\{-1,0,1\}]$.

The evaluation operator E: $P_I[\beta,D] \to Z$ maps radix polynomials to their integer values. For a base β, the digit set D is

(i) complete for β if E: $P_I[\beta,D] \to Z$ is onto Z,

(ii) non-redundant for β if E: $P_I[\beta,D] \to Z$ is one-to-one to **Z**,

(iii) basic for β if E: $P_I[\beta,D] \to Z$ is a one-to-one correspondence.

The digit set $\{-1,0,1\}$ is complete but not basic for base 2, and the standard digit set $\{0,1,2,\ldots,\beta-1\}$ is non-redundant but not basic for base $\beta \geq 2$, since no negative numbers are representable in the latter system. The standard digit set $\{0,1,2,\ldots,|\beta|-1\}$ for the negative base $|\beta| \leq -2$ and the digit set $\{-1,0,1\}$ for base 3 provide examples of the basic digit set and base pairs that we seek to characterize.

A complete residue system modulo μ is a set $S \subset Z$ with $\mu = |S| \geq 2$ where S contains exactly one integer $s_i \in S$ with $s_i \equiv i \bmod \mu$ for each i, $0 \leq i \leq \mu - 1$.

Theorem 1: Let D be a basic digit set for the base β. Then D is a complete residue system modulo the absolute value, $|\beta|$, of the base.

Proof: Assume D is basic for β. For $0 \leq i \leq |\beta|-1$, there exists $d_m[\beta]^m + d_{m-1}[\beta]^{m-1} +\ldots+ d_1[\beta] + d_0 \in P_I[\beta,D]$ of value i, hence $d_0 \equiv i \bmod |\beta|$, and D contains a complete residue modulo $|\beta|$. Let

$d' \equiv d'' \bmod |\beta|$ for some $d',d'' \in D$. Then $j = (d'-d'')/\beta$ is an integer, so there exists $P_j \in P_I[\beta,D]$ of value j. It follows that $P_j \times [\beta] + d''$ and d' are both members of $P_I[\beta,D]$ of value d', and they must be identical since D is basic for β. So $P_j = 0$, $d' = d''$, and D is a complete residue system modulo $|\beta|$. ▌

Note that $D = \{-2,0,2\}$ is a complete residue system modulo 3 which is not basic for base 3, since $P_I[3,\{-2,0,2\}]$ contains only even valued radix polynomials. A weaker converse is obtained.

Lemma 2: Let β be a base and D a complete residue system modulo $|\beta|$ with $0 \in D$. Then D is a non-redundant digit set for base β.

Proof: Assume the distinct radix polynomials $P,Q \in P_I[\beta,D]$ have the same value. For $P = \sum_{i=0}^{\infty} d_i[\beta]^i$, $Q = \sum_{i=0}^{\infty} a_i[\beta]^i$, where only a finite number of d_i and a_i are non-zero, and $d_i,a_i \in D$, let $k = \min\{i \mid d_i \neq a_i\}$. Then

$$\sum_{i=k}^{\infty} d_i\beta^i = \sum_{i=k}^{\infty} a_i\beta^i.$$

Multiplying by β^{-k} and considering residues modulo $|\beta|$,

$$d_k \equiv a_k \bmod |\beta| .$$

But $d_k \neq a_k$, d_k, $a_k \in D$, which contradicts the assumption that D is a complete residue system modulo $|\beta|$. Thus D is a non-redundant digit set for β. ▌

The study of basic digit sets thus reduces to the determination of those complete residue systems which are complete digit sets. For the complete residue system D modulo $|D|$, the residue of i in D is denoted by $\|i\|_D$ and defined uniquely by

$$\|i\|_D = i + k|D| \qquad \text{for some } k \in Z,$$
$$\|i\|_D \in D. \tag{3}$$

For the base β and complete residue system D modulo $|\beta|$, the base β chop function Φ: $Z \to Z$ is defined by

$$\Phi(i) = (i-\|i\|_D)/\beta \qquad \text{for } i \in Z. \tag{4}$$

The n-place base β chop function Φ^n: $Z \to Z$ is given for $n \geq 0$ by

$$\Phi^0(i) = i,$$
$$\Phi^n(i) = \Phi(\Phi^{n-1}(i)) \qquad \text{for } n \geq 1. \tag{5}$$

The chop function is defined on the integers, but its important implications for radix representation are stated in the following lemma which is an immediate consequence of the definitions (4), (5).

Lemma 3: Let β be a base and D a digit set which is a complete residue system modulo $|\beta|$. Let $d_m[\beta]^m + d_{m-1}[\beta]^{m-1} +\ldots+ d_1[\beta]+d_0 \in P[\beta,D]$ have value i. Then

$$\Phi^k(i)=d_m\beta^{m-k}+d_{m-1}\beta^{m-k-1}+\ldots+d_{k+1}\beta+d_k \quad \text{for } 0 \leq k \leq m,$$
$$\Phi^{m+1}(i) = 0. \tag{6}$$

The operation of the chop function is illustrated for the balanced ternary number $1\,\bar{1}\,0\,1\,1\,\bar{1}\,\bar{1}_3 = 518$.

Φ^0 (518)	= 518	= 518	$= 1\,\bar{1}\,0\,1\,1\,\bar{1}\,\bar{1}_3$
Φ^1 (518)	= (518-(-1))/3	= 173	$= 1\,\bar{1}\,0\,1\,1\,\bar{1}_3$
Φ^2 (518)	= (173-(-1))/3	= 58	$= 1\,\bar{1}\,0\,1\,1_3$
Φ^3 (518)	= (58-1)/3	= 19	$= 1\,\bar{1}\,0\,1_3$
Φ^4 (518)	= (19-1)/3	= 6	$= 1\,\bar{1}\,0_3$
Φ^5 (518)	= (6-0)/3	= 2	$= 1\,\bar{1}_3$
Φ^6 (518)	= (2-(-1))/3	= 1	$= 1_3$
Φ^7 (518)	= (1-1)/3	= 0	

Lemma 4: Let β be a base and D a complete residue system modulo $|\beta|$ with $0 \in D$. Then for any $i \in Z$ either there is a minimum m such that $\Phi^n(i) = 0$ for $n \geq m$ or there are minimal $t,p \in Z$ such that $\Phi^t(i) = \Phi^{t+p}(i) \neq 0$.

Proof: For $|i| > \max\{|d| \mid d \in D\}$, it follows from (4) that $|\Phi(i)|<|i|$. The sequence of integers $\Phi^0(i)$, $\Phi^1(i)$, $\Phi^2(i)$,... thus has at most a finite number of distinct values in its terms, and the lemma follows. ▌

For the base β and complete residue system D modulo $|\beta|$ with $0 \in D$, the base β degree of i in D for any $i \in Z$ is denoted by deg(i) where

$$\deg(i) = \begin{cases} -\infty & \text{if } i=0, \\ \min\{n \mid \Phi^{n+1}(i)=0\} & \text{if } i\neq 0 \text{ and } \Phi^m(i) = 0 \text{ for some } m, \\ \infty & \text{otherwise.} \end{cases} \tag{7}$$

Furthermore Φ cycles for i with period p if $i \neq 0$ and $p = \min\{n \mid \Phi^n(i) = i, n \geq 1\}$. If Φ cycles for i with period p for some $i,p \in Z$, then Φ is cyclic, otherwise Φ is acyclic.

Example 1:

a) $\beta = -3$, $D = \{-1,0,91\}$, $i = -12$

$$\phi^1(-12) = \phi(-12) = (-12-0)/-3 = 4$$
$$\phi^2(-12) = \phi(4) = (4-91)/-3 = 29$$
$$\phi^3(-12) = \phi(29) = (29-(-1))/-3 = -10$$
$$\phi^4(-12) = \phi(-10) = (-10-(-1))/-3 = 3$$
$$\phi^5(-12) = \phi(3) = (3-0)/-3 = -1$$
$$\phi^6(-12) = \phi(-1) = (-1-(-1))/-3 = 0$$

Thus $\deg(-12) = 5$, and the radix polynomial $-[-3]^5-[-3]^3-[-3]^2+91[-3]$ of $\mathcal{P}_I[-3,\{-1,0,91\}]$ has value -12.

b) $\beta = 3$, $D = \{-1,0,91\}$, $i = -5$

$$\phi^1(-5) = \phi(-5) = (-5-91)/3 = -32$$
$$\phi^2(-5) = \phi(-32) = (-32-91)/3 = -41$$
$$\phi^3(-5) = \phi(-41) = (-41-91)/3 = -44$$
$$\phi^4(-5) = \phi(-44) = (-44-91)/3 = -45$$
$$\phi^5(-5) = \phi(-45) = (-45-0)/3 = -15$$
$$\phi^6(-5) = \phi(-15) = (-15-0)/3 = -5$$

Thus ϕ cycles for -5 with period 6.

Theorem 5: For the base β, let the digit set D be a complete residue system modulo $|\beta|$, and let $i \in Z$, $i \neq 0$. Then either

(i) $\deg(i)=n$ and

$$i=\|\phi^n(i)\|_D\beta^n+\|\phi^{n-1}(i)\|_D\beta^{n-1}+\ldots+\|\phi^1(i)\|_D+\|i\|_D, \tag{8}$$

or

(ii) there are minimal t,p such that $\phi^t(i) = \phi^{t+p}(i)\neq 0$, and with $j=\phi^t(i)$,

$$-j(\beta^p-1)=\|\phi^{p-1}(j)\|_D\beta^{p-1}+\|\phi^{p-2}(j)\|_D\beta^{p-2}+\ldots+\|\phi^1(j)\|_D\beta+\|j\|_D. \tag{9}$$

Proof: Let $i \in Z$, $i \neq 0$, so from (4)
$$i = \phi(i)\,\beta + \|i\|_D.$$

Applying the same formula to $\phi(i)$ yields
$$\phi(i) = \phi^2(i)\,\beta + \|\phi(i)\|_D$$
so that by substitution
$$i = \phi^2(i)\,\beta^2 + \|\phi(i)\|_D\,\beta + \|i\|_D,$$
and continuing with substitutions of $\phi^k(i) = \phi^{k+1}(i)\beta+\|\phi^k(i)\|_D$,
$$i=\phi^{n+1}(i)\beta^{n+1}+\|\phi^n(i)\|_D\beta^n+\ldots+\|\phi^1(i)\|_D\beta+\|i\|_D$$
$$\text{for any } n \geq 0. \tag{10}$$

Thus if $\deg(i) = n$, $\phi^{n+1}(i) = 0$ and equation (8) for i is established. If $\deg(i) \neq n$ for any finite n, then by Lemma 4 there exist minimal t,p such that $\phi^t(i) = \phi^{t+p}(i) \neq 0$. Letting $j = \phi^t(i)$, application of (10) to j yields
$$j=\phi^p(j)\beta^p+\|\phi^{p-1}(j)\|_D\beta^{p-1}+\ldots+\|\phi^1(j)\|_D\beta+\|j\|_D,$$
and since $\phi^p(j) = j$, equation (9) follows. ∎

From Theorems 1 and 5 and Lemmas 2 and 3 we obtain the following.

Corollary 5.1: D is a basic digit set for the base β iff D is a complete residue system modulo $|\beta|$ with $0 \in D$ such that $\deg(i)$ is finie for all non-zero $i \in Z$, i.e. iff ϕ is acyclic.

When ϕ cycles for i with period p, then (9) may be applied to each term $\phi^k(i)$, $0 \leq k \leq p-1$, of the cycle, and the following is obtained.

Corollary 5.2: For the base β, let the digit set D be a complete residue system modulo $|\beta|$. Suppose ϕ cycles for $i \in Z$, $i \neq 0$ with period p. Then

$$-\sum_{k=0}^{p-1}\phi^k(i) = \sum_{k=0}^{p-1}\|\phi^k(i)\|_D/(\beta-1). \tag{11}$$

For the base β, note that if D is a complete residue system modulo $|\beta|$ with $k(\beta-1) \in D$ for $k \neq 0$, then ϕ cycles for $-k$ with period 1, and D is not a basic digit set for base β. This yields a second fundamental condition for the digit set D to be basic for base β.

Theorem 6: Let D be a basic digit set for base β. Then D contains no digit of value $k(\beta-1)$ for any $k \neq 0$.

For the digit set D, base β, and $n \geq 1$, let the n-place digit set D^n be given by

$$D^n=\{i\,|\,i=d_{n-1}\beta^{n-1}+d_{n-2}\beta^{n-2}+\ldots+d_1\beta+d_0,\ d_j \in D \text{ for } 0 \leq j \leq n-1\}. \tag{12}$$

If D is a basic digit set for base β, then by considering blocks of n term length in a radix polynomial $P \in P_I[\beta,D]$, it is evident that D^n is a basic digit set for base β^n for every $n \geq 1$. Then the simple necessary conditions of Theorem 1 and Theorem 6 must apply to every member D^n, β^n for $n \geq 1$ of this family of basic digit set and base pairs. Our principal result is that these conditions are also sufficient to verify that D is basic for β.

Theorem 7 (Characterization Theorem for Basic Digit Sets):

D is a basic digit set for base β iff D is a complete residue system modulo $|\beta|$ with $0 \in D$ where the n-place digit set D^n given by (12) contains no nonzero multiple of β^n-1 for any $n \geq 1$.

Proof: If D is a basic digit set for base β, then D^n is basic for base β^n for every $n \geq 1$ and the conditions follow from Theorem 1 and Theorem 6.

Conversely, for the base β, suppose D is a complete residue system modulo $|\beta|$ with $0 \in D$ where D^n contains no non-zero multiple of β^n-1 for any $n \geq 1$. Then by Theorem 5, Φ cannot cycle for any $i \neq 0$, and by Corollary 5.1, D is then basic for base β. |

III. Complexity of Basic Digit Set Verification and Radix Conversion

The theoretical characterization of basic digit sets given by Theorem 7 does not yield an efficient computational procedure for comfirming that a given digit set D is basic for base β. We now show that basic digit set verification can be reduced to determining that Φ does not cycle for i for a particular small interval of values of i.

Lemma 8: For the base β, let the digit set D be a complete residue system modulo $|\beta|$. Let $d_{min} = \min\{d|d \in D\}$ and $d_{max} = \max\{d|d \in D\}$. Then Φ can cycle for i only for values of i in the interval

$$\text{(i)}\quad \frac{-d_{max}}{\beta-1} \leq i \leq \frac{-d_{min}}{\beta-1} \quad \text{if } \beta = |D|,$$

$$\text{(ii)}\quad \frac{-d_{min}\beta-d_{max}}{\beta^2-1} \leq i \leq \frac{-d_{max}\beta-d_{min}}{\beta^2-1} \quad \text{if } \beta = -|D|. \tag{13}$$

Proof: Consider the positive base case, $\beta = |D|$. For $i > -d_{min}/(\beta-1)$,

$$\Phi(i) \leq \frac{i-d_{min}}{\beta} < \frac{i+(\beta-1)i}{\beta} = i,$$

and for $i \geq -d_{max}/(\beta-1)$,

$$\Phi(i) \geq \frac{i-d_{max}}{\beta} \geq \frac{-d_{max}-(\beta-1)d_{max}}{(\beta-1)\beta} = -\frac{d_{max}}{\beta-1}.$$

Similarly $i < -d_{max}/(\beta-1)$ implies $\Phi(i) > i$, and $i \leq -d_{min}/(\beta-1)$ implies $\Phi(i) \leq -d_{min}/(\beta-1)$. These inequalities imply that Φ can cycle for i only if $-d_{max}/(\beta-1) \leq i \leq -d_{min}/(\beta-1)$ for $\beta = |D|$, which verifies the positive base condition (13) (i) of the lemma.

Now consider the negative base case, $\beta = -|D|$. Φ is acyclic iff D is basic for $\beta = -|D|$ iff the 2-place digit set D^2 given by (12) is basic for β^2, for which the positive base condition applies. Since $\min\{d|d \in D^2\} = d_{max}\beta+d_{min}$ and $\max\{d|d \in D^2\} = d_{min}\beta+d_{max}$, condition (ii) follows from condition (i) applied to the digit set D^2 for base β^2. |

Lemma 8 is sharp in that if $\beta = |D|$ and $d_{max} = k(\beta-1)$ and/or if $d_{min} = j(\beta-1)$, then Φ cycles for $-d_{max}/(\beta-1)$ and/or $-d_{min}/(\beta-1)$, **respectively**. For $\beta = -|D|$, if $d_{max} = -d_{min} = -k(\beta+1)$, then Φ cycles for both $-k$ and k, and if $d_{max} = \beta^2-1$, $d_{min} = 0$, then Φ cycles for $-\beta$ and -1.

From lemma 8 it is possible to construct an efficient procedure to determine if D is basic for β.

Corollary 8.1: For the base β, let the digit set D be a complete residue system modulo $|\beta|$. Then D may be determined to be basic for β or not in at most $(\max\{d|d \in D\} - \min\{d|d \in D\})/(|\beta|-1)$ applications of Φ.

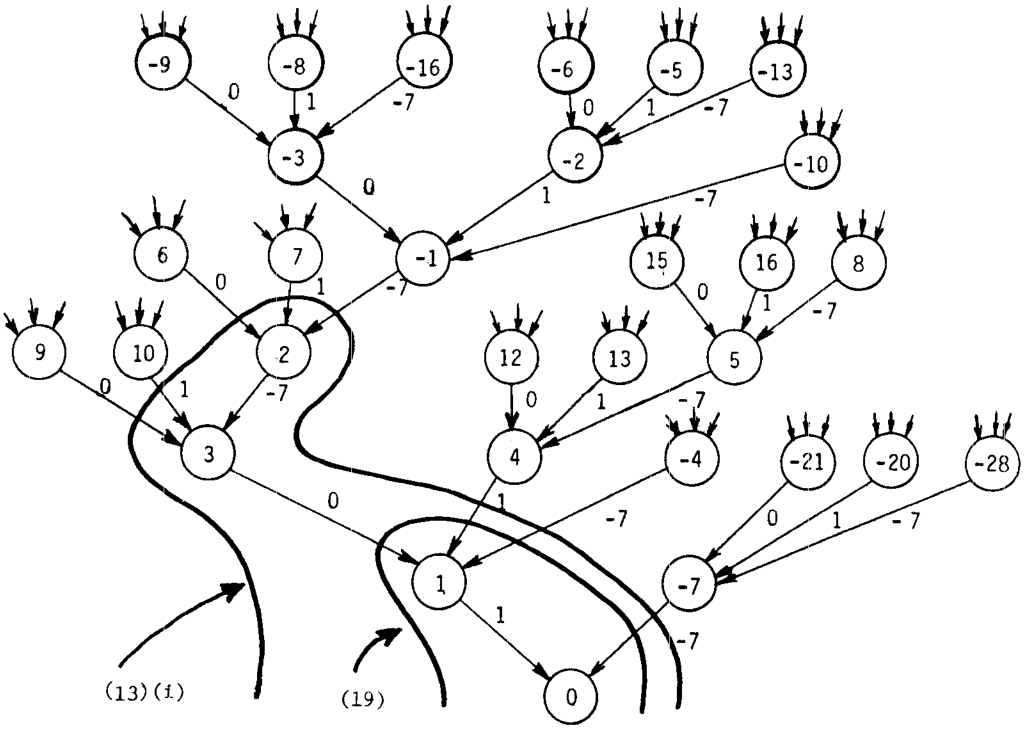
Proof: Recursively select an unevaluated i in the interval specified by (13)(i) for $\beta = |D|$ or (13)(ii) for $\beta = -|D|$ and evaluate $\Phi^k(i)$, $k = 1,2,\ldots$, until $\Phi^k(i)$ yields zero, a repeat value $\Phi^k(i) = \Phi^j(i)$ for $j < k$ determining a cycle, or a value known to lead to zero. This procedure methodically either determines a cycle or proves Φ to be acyclic by evaluating Φ at most at every non-zero value of i in the interval specified by (13)(i) or (13)(ii). Since both intervals have $(\max\{d|d \in D\} - \min\{d|d \in D\})/(|\beta|-1)$ non-zero integral values, the corollary is obtained. |

An appropriate structure for illustrating the computation of radix representation and basic digit set verification is a labeled directed graph. For a base β and digit set D which is a complete residue system modulo $|\beta|$, the digital digraph is the directed graph with the integers as vertices where there is a directed edge from i to $\Phi(i)$ with label $\|i\|_D$ for every $i \neq 0$.

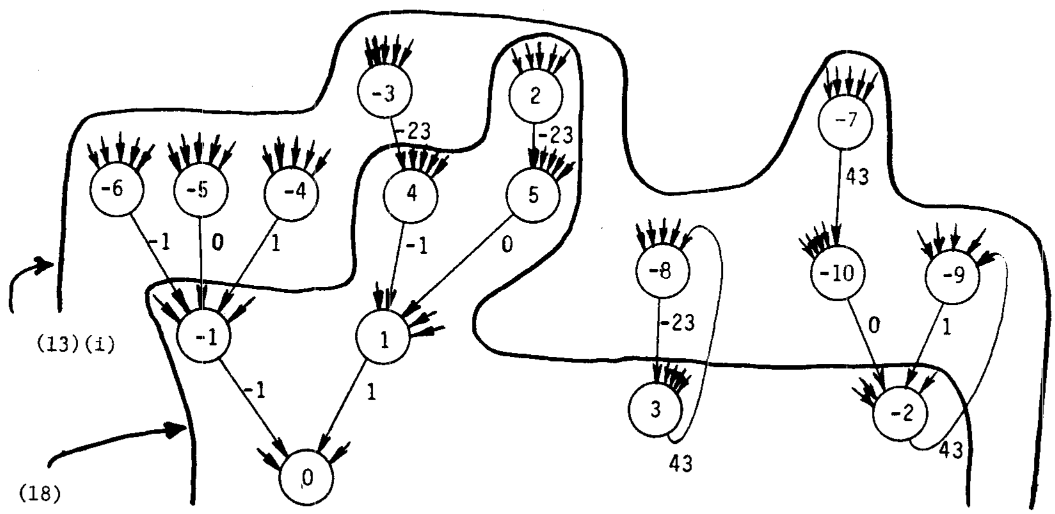
Example 2:

a) For $\beta = 3$, $D = \{0,1,-7\}$, Figure 1a shows a portion of the digital digraph. The members of the interval (13)(i) are noted, as are the members of the subinterval (19) which is shown by Lemma 13 of the next section to contain at least one member of any cycle of Φ. The fact that all vertices within the (13)(i) bound are connected to vertex 0 confirms that D is basic for 3.

b) For $\beta = 5$ and $D = \{0,1,-23,43,-1\}$, Figure 1b shows the portion of the digital digraph containing all vertices of the interval $-10,-9,\ldots,5$ indicated by (13)(i). Note that the members $-2,-1,\ldots,5$ indicated by (18) intersect all cycles for Φ.



a) $\beta = 3$, $D = \{0,1,-7\}$



b) $\beta = 5$, $D = \{0,1,-23,43,-1\}$

Figure 1: Portions of the digital digraph for a) $\beta = 3$, $D = \{0,1,-7\}$, and b) $\beta = 5$, $D = \{0,1,-23,43,-1\}$, illustrating the important intervals characterized by formulas (13)(i), (18) and (19).

The digital digraph has indegree $|\beta|$ and outdegree unity for every non-zero vertex i. The set of vertices at distance no greater than n from vertex 0 constitute the n-place digit set D^n as defined by (12). Thus $D^2 = \{0,1,-7,3,4,-4,-21,-20,-28\}$ for $D = \{0,1,-7\}$ as seen in Figure 1a. Finally, the radix representation of i in positional notation is derived from the digital digraph by concatenating the edge labels on the path from i to 0 in right to left order, e.g. $-9_{10} = 10\bar{7}\bar{7}00_3 = 1 \times 3^5 - 7 \times 3^3 - 7 \times 3^2$ from Figure 1a, and $2_{10} = 1,0,\overline{23}_5 = 1 \times 5^2 - 23$ from Figure 1b.

Radix conversion is the process of determining $P_i \in P_I[\beta,D]$ of value i when such a P_i exists. If the digit set D is a complete residue system modulo $|\beta|$, it is sufficient by Theorem 5 to apply Φ recursively deg(i) + 1 times to determine $P_i \in P_I[\beta,D]$. The following bound on deg(i) in terms of β and D applies to all finite deg(i), and thus implicitly bounds the complexity of determining if P_i exists.

Lemma 9: For the base β, let the digit set D be a complete residue system modulo $|\beta|$. If $i \neq 0$ and deg(i) is finite, then with $\Delta = \max\{|d| \mid d \in D\}$,

$$\frac{\log|i|}{\log|\beta|} - \frac{\log \Delta}{\log|\beta|} - 1 \leq \deg(i) \leq \frac{\log|i|}{\log|\beta|} + \frac{2\Delta}{|\beta|-1} + 1. \quad (14)$$

Proof: For $i \neq 0$ with $|\beta|^{m-1} < |i| < |\beta|^m$, recursive application of (4) yields

$$\begin{aligned} |\phi^m(i)| &\leq \Delta/|\beta| + |\phi^{m-1}(i)|/|\beta| \\ &\leq \Delta(|\beta|^{-1} + |\beta|^{-2} + \ldots + |\beta|^{-m}) + |i|/|\beta|^m \\ &\leq \Delta/(|\beta|-1) + 1. \end{aligned} \quad (15)$$

It follows from (4) that $|\Phi(j)| \leq |j|-1$ whenever $|j| > \Delta/(|\beta|-1)$, and $|\Phi(j)| \leq \Delta/(|\beta|-1)$ whenever $|j| \leq \Delta/(|\beta|-1)$, so then $|\Phi^k(i)| \leq \Delta/(|\beta|-1)$ for all $k \geq m+1$. Thus the sequence $\Phi^1(i), \Phi^2(i), \ldots, \phi^k(i), \ldots$ must either reach zero or a repeat value for $k \leq m + 1 + 2\Delta/(|\beta|-1)$, so assuming deg(i) is finite,

$$\deg(i) \leq \frac{\log|i|}{\log|\beta|} + \frac{2\Delta}{|\beta|-1} + 1.$$

For $|i| \leq \Delta$, (14) holds, so assume $|i| > \Delta$ and choose n maximum so that

$$|i| > \Delta\ (|\beta|^{n-1} + |\beta|^{n-2} + \ldots + 1).$$

Then from (4),

$$|\Phi(i)| > \Delta\,(|\beta|^{n-2} + |\beta^{n-3}| + \ldots + 1),$$

$$|\Phi^{n-1}(i)| > \Delta.$$

Hence $\deg(\Phi^{n-1}(i)) \geq 1$, so $\deg(i) \geq n$. Furthermore since $|i| \leq |\beta|^{n+1}\Delta$, it follows that $n + 1 \geq (\log|i| - \log\Delta)/\log|\beta|$, completing the lemma. |

Corollary 9:1: For the base β, let D be a digit set which is a complete residue system modulo $|\beta|$, and let $i \in Z$, $i \neq 0$. Then after at most $\lfloor \log|i|/\log|\beta| + 2\Delta/(|\beta|-1) + 2 \rfloor$ iterative applications of Φ to i either the unique $P_i \in P_I[\beta,D]$ of value i is determined or the non-existence of any $P \in P_I[\beta,D]$ of value i is confirmed.

Proof: The result is immediate from Theorem 5 and Lemma 9. |

Thus the determination of the particular radix polynomial for representing $i \in Z$ can be accomplished with complexity $O(\frac{\Delta}{|\beta|} + \frac{\log|i|}{\log|\beta|})$, where $\Delta = \max\{|d| \mid d \in D\}$.

IV. Classes of Basic Digit Sets

There are no basic digit sets for $\beta = 2$. $D = \{0,1\}$ and $D = \{0,-1\}$ are the only basic digit sets for $\beta = -2$. A digit set D is termed normal for base β if $\max\{|d| \mid d \in D\} \leq |\beta|-1$. The normal basic digit sets are readily characterized.

Lemma 10: For the base β, let the normal digit set D be a complete residue system modulo $|\beta|$. Then D is basic for β iff

$$\begin{aligned} &\text{(i)} \quad \{-1,1\} \subset D && \text{for } \beta = |D|, \\ &\text{(ii)} \quad -1 \in D \text{ or } 1 \in D \text{ (or both)} && \text{for } \beta = -|D|. \end{aligned} \quad (16)$$

Proof: For any normal digit set D for base β, it is sufficient by Lemma 8 simply to verify that there exist $P_i \in P_I[\beta,D]$ of value i for i = -1,0,1. For $\beta = |D|$, Theorem 6 requires $\beta-1 \notin D$, $-\beta+1 \notin D$, so condition (i) is necessary and sufficient. For $\beta = -|D|$, note that if neither -1 nor 1 were in D, then $\Phi(-1) = 1$, $\Phi(1) = -1$, and ϕ cycles for -1 and 1. If either $-1 \in D$, $1 \in D$, or $\{-1,1\} \subset D$, then Φ does not cycle for either -1 or 1, verifying (ii). |

If the digit set D is normal for base β, then there are only two possible digit values for each non-zero residue in choosing D to be a complete residue system modulo $|\beta|$, and the following is immediate.

Corollary 10.1: For $|\beta| \geq 3$, there are $2^{|\beta|-3}$ normal basic digit sets for the positive base $\beta \geq 3$, and $3 \times 2^{|\beta|-3}$ normal basic digit sets for the negative base $\beta \leq -3$.

Results of de Bruijn on binary based "good pairs" effectively establish that there are infinite classes of basic digit sets for base 4 when the digit values are allowed to be larger than the base [1]. The following theorem characterizes an infinite class of basic digit sets for any base $\beta \geq 3$.

Theorem 11: For any $n \geq 1, \beta \geq 3$, the digit set

$$D(\beta,n) = \{0,1,2,\ldots,\beta-3,\ \beta-2,\ (-\beta^n+\beta-1)\} \qquad (17)$$

is basic for base β.

Proof: Every positive integer $i < \beta^n - 1$ is the value of a standard base β radix polynomial

$$P_i = d_m[\beta]^m + d_{m-1}[\beta]^{m-1} + \ldots + d_1[\beta] + d_o$$

where $0 \leq d_j \leq \beta-1$ for $0 \leq j \leq m$, and $\deg(P_i)=m \leq n-1$. Replacing each term $d_k[\beta]^k$ above for which $d_k=\beta-1$ by $1 \times [\beta]^{k+n} + (-\beta^n+\beta-1) \times [\beta]^k$, we derive a radix polynomial P_i^* having all digit values in $D(\beta,n)$ where $\deg(P_i^*) \leq m + n \leq 2n-1$ and P_i^* also has value i for all $1 \leq i \leq \beta^n - 1$. From Lemma 8 it follows that $D(\beta,n)$ is basic for β for any $\beta \geq 3$ and any $n \geq 1$. |

For any negative base $\beta \leq -3$ and standard digit set $D_{|\beta|} = \{0,1,\ldots,|\beta|-1\}$, it is readily verified that there is a standard negative base radix polynomial $P_i \in P[\beta,D_{|\beta|}]$ of value i with $\deg(P_i) \leq 2k-1$ whenever $-2|\beta|^{2k-1} \leq i \leq 2|\beta|^{2k-2}$ for $k \geq 1$. Letting $D^* = \{0,1,2,\ldots,|\beta|-3,|\beta|-2,\ (-|\beta|^{2k}-|\beta|-1)\}$, then for any i, $-2|\beta|^{2k-1} \leq i \leq 2|\beta|^{2k-2}$, proceeding as in the proof of Theorem 11 a radix polynomial $P_i^* \in P[\beta,D^*]$ of value i is then shown to exist, which by Lemma 8, proves the following.

Theorem 12: For any $\beta \leq -3$, $k \geq 1$, the digit set

$$D^* = \{0,1,2,\ldots,|\beta|-3,|\beta|-2,(-|\beta|^{2k}+|\beta|-1)\}$$

is basic for base β.

For the base β and digit set D, the interval specified by (13) must contain all integers for which Φ is cyclic. If D contains no non-zero multiples of $\beta-1$, then any cycle for Φ must have period at least two. This observation may be exploited to yield a subinterval which must contain at least one element for which Φ cycles whenever Φ is cyclic.

Lemma 13: Let the digit set D be a complete residue system modulo $\beta = |D| > 3$ without non-zero multiples of $\beta - 1$. Let $t_1 = \max\{d|d \in D\}$, $t_2 = \max\{d|d \in D, d \neq t_1\}$, and $d_{min} = \min\{d|d \in D\}$. Then D is basic for β iff there exists $P_j \in P_I[\beta,D]$ of value j for all

$$-\left(\frac{t_1}{\beta(\beta-1)} + \frac{t_2}{\beta}\right) \leq j \leq -\frac{d_{min}}{\beta-1}\ . \qquad (18)$$

Proof: Suppose Φ cycles for i with period p. From Theorem 5,

$$-i(\beta^p-1) = \|\Phi^{p-1}(i)\|_D\beta^{p-1} + \|\Phi^{p-2}(i)\|_D\beta^{p-2} + \ldots + \|\Phi^1(i)\|_D\beta + \|i\|_D.$$

If $\|\Phi^k(i)\|_D = t_1$ for $0 \leq k \leq p-1$, then $-i(\beta^p-1) = t_1(\beta^p-1)/(\beta-1)$ and t_1 is a non-zero multiple of $\beta-1$, a contradiction. Hence $p \geq 2$ and $\|\Phi^k(i)\|_D \leq t_2$ for some k. Then Φ cycles for $j = \Phi^{k+1}(i)$ and

$$-j(\beta^p-1) \leq t_2\beta^{p-1} + t_1(\beta^{p-2}+\beta^{p-3}+\ldots+1)$$
$$\leq t_2(\beta^p-1)/(\beta-1) + (t_1-t_2)(\beta^{p-1}-1)/(\beta-1)$$

so then

$$-j < \frac{t_2}{\beta-1} + \frac{t_1-t_2}{\beta(\beta-1)} = \frac{t_1}{\beta(\beta-1)} + \frac{t_2}{\beta}\ ,$$

and by Lemma 8, the proof is complete. |

In like manner one obtains the result of Lemma 13 with the interval specified by (18) replaced by

$$-\frac{d_{max}}{\beta-1} \leq j \leq -\left(\frac{s_1}{\beta(\beta-1)} + \frac{s_2}{\beta}\right)\ , \qquad (19)$$

where s_1 and s_2 are the smallest and second smallest elements of D, respectively, and $d_{max}=\max\{d|d \in D\}$.

The determination that D is basic for $\beta \geq 3$ can be accomplished by showing that there exists $P_i \in P_I[\beta,D]$ of value i for all i in the interval specified by (18) or (19). Lemma 13 is of greatest assistance when the interval specified by (18) or (19) is a subset of D and D contains no non-zero multiple of $\beta-1$, for then D is immediately confirmed to be basic for $\beta = |D|$.

Example 3:

For base 7, the digit set $D = \{0,1,9,52,-10,-2,-1\}$ is a complete residue system modulo 7. Lemma 8 would require computing Φ for $\{-8,-7,-6,-5,-4,-3,-2,-1,0,1\}$ to determine that D is basic for 7. By Lemma 13, (18) yields $\{-2,-1,0,1\}$ D, and since D has no non-zero multiple of 6, D is basic for 7.

Lemma 13 may be utilized to derive numerous classes of basic digit sets. The following corollary is stated without proof to indicate the nature of the construction. A proof can be fashioned similar to the methodology of the proof of Theorem 11.

Corollary 13.1: Let D be a basic digit set for base $\beta \geq 4$ with $\Delta = \max\{|d| \mid d \in D\}$, where $j \in D$ for all j such that $|j| \leq 1 + \Delta/(\beta-1)$. For a fixed $d' \in D$, $d' \neq 0$, $d' \not\equiv -1 \bmod(\beta-1)$, and any $k \geq 3$, let S_k be the digit set formed from D by replacing d' with $d' + \beta^k$. Then S_k is basic for β for all $k \geq 3$.

Example 4:

Let $D = \{0,1,2,3,14,25,26,-3,-2,-1\}$ and $\beta = 10$. Then $1+\Delta/(\beta-1) = 1 + 26/(10-1) = 3\frac{8}{9}$. So from Corollary 13.1, $\{0,1,2,3,14,(10^k+25),26,-3,-2,-1\}$ is basic for base 10 for any $k \geq 3$.

An interesting class of digit sets for base 3 are those of the form $D_k = \{0,1,-6k-1\}$. From (19) it is observed that D_k is basic for base 3 iff there exists $P_i \in P_I[\beta,D]$ of value i for $0 \leq i \leq k$. Table 1 shows those D_k which are basic and those Φ that are cyclic for $0 \leq k \leq 14$, and no clearly identifiable pattern for basic D_k in terms of k is observable. Note that k = 0,1,4, and 13 yield D_k which are basic for 3 by theorem 11.

k	D_k	Basic for 3	Cycle for Φ
0	{0,1,-1}	Yes	
1	{0,1,-7}	Yes	
2	{0,1,-13}	No	$2 \to 5 \to 6 \to 2$
3	{0,1,-19}	No	$2 \to 7 \to 2$
4	{0,1,-25}	Yes	
5	{0,1,-31}	Yes	
6	{0,1,-37}	Yes	
7	{0,1,-43}	No	$5 \to 16 \to 5$
8	{0,1,-49}	No	$2 \to 17 \to 22 \to 7 \to 2$
9	{0,1,-55}	No	$2 \to 19 \to 6 \to 2$
10	{0,1,-61}	No	$2 \to 21 \to 7 \to 2$
11	{0,1,-67}	No	$8 \to 25 \to 8$
12	{0,1,-73}	Yes	
13	{0 1,-79}	Yes	
14	{0,1,-85}	Yes	

Tabel 1: Digit Sets $D_k = \{0,1,-6k-1\}$ for $k = 0,1,2,\ldots, 14$, showing those D_k that are basic for 3 and a cycle for Φ when D_k is not basic.

V. Some Results on Radix Representation of the Reals

For the base β and digit set D, a finite precision base β radix polynomial over D is either the zero polynomial or an extended polynomial expression over Z in the constant β,

$$P([\beta]) = d_m[\beta]^m + d_{m-1}[\beta]^{m-1} + \ldots + d_\ell[\beta]^\ell, \qquad (20)$$

where $d_i \in D \subset Z$ for $-\infty < \ell \leqq i \leqq m < \infty$, and $d_m \neq 0$, $d_\ell \neq 0$. The radix representation system $P[\beta,D]$ is the set of all such finite precision base β radix polynomials over D. An infinite precision base β radix polynomial over D is given by the extended polynomial expression

$$P([\beta]) = d_m[\beta]^m+d_{m-1}[\beta]^{m-1}+\ldots+d_0+d_{-1}[\beta]^{-1}+\ldots \qquad (21)$$

where $d_i \in D \subset Z$ for $i \leqq m$, and $d_i \neq 0$ for m and infinitely many indices $i < m$. The infinite precision radix representation system $P_\infty[\beta,D]$ is the set of all finite and infinite precision radix polynomials over D.

The details of radix representation of the reals are beyond the scope of this paper. Our further research has shown many properties for such systems, and the following summary relevant to definitions (20) and (21) indicate some of our major results.

When D is a bisic digit set for base β,

(a) the integer radix representation system $P_I[\beta,D]$ is complete and non-redundant for the integers Z,

(b) the finite precision radix representation system $P[\beta,D]$ is complete and non-redundant for the $|\beta|$-ary numbers $A_{|\beta|} = \{i|\beta|^j \mid i,j \in Z\}$,

(c) the infinite precision radix representation system $P_\infty[\beta,D]$ is complete for the reals and redundant for a set S of reals disjoint from $A_{|\beta|}$, where S is at least countable and in some cases uncountable, and where each member of S may be the value of strictly more than two but never more than $\lfloor 2\max\{|d| \mid d \in D\}/(|\beta|-1)\rfloor + 1$ members of $P_\infty[\beta,D]$.

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