SYSTEMS OF NUMERATION

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Abstract

A numeration system is a set of integers (basis elements) such that every integer can be represented uniquely over the set using integer digits of bounded size. Such systems are scattered in many fields in mathematics and computer science. Many of the known ones and new ones are unified and derived from a basic result on recursively defined basis elements. Applications are indicated.

1. Introduction

There are many ways of representing an integer uniquely! The best known method is the decimal system. Whereas the Maya Indians used base 20 (using the fingers on hands and feet), some of the human race became recently more primitive using the binary system instead, being influenced by the computer race which, for electronic reasons, is zealously addicted to the binary system. It may be of interest to computers to know that there are actually infinitely many binary systems!

Somewhat less known systems of numeration include mixed radix, factorial representation, and exotic systems based on recurrence relations, a special case of which is the Fibonacci system of numeration. So there are many ways of representing an integer uniquely; many ways, that is, in each of which an integer can be represented uniquely.

These and other systems of numeration normally hide in various unexpected places, where they are applied for varied purposes. Typically, when the need for a numeration system arises, it is defined and an ad hoc proof of its capability to represent integers uniquely is given. The purpose of this article is to unify these results and show how they can be derived simply and uniformly.

A very simple yet general system of numeration is presented in Theorem 1. It may be used to derive all the numeration systems we intend to present, but some repetitive argumentation is involved. We prefer instead to use Theorem 1 to derive a general numeration system based on recursively defined basis elements. This is done in Theorem 2, which sheds more light on the nature of numeration systems than Theorem 1. Our numeration systems are then derived from Theorem 2. All but one. The exceptional system is based on a recurrence relation with a negative coefficient, whereas the recurrence relations of Theorem 2 contain only positive coefficients. The exceptional system is therefore derived directly from Theorem 1.

Theorems 1 and 2 are given in Section 2. The derivation of the numeration systems from Theorem 2 is carried out in Section 3 in a rather slicker way than Theorem 1 would permit. The exceptional system is derived in the final Section 4. Applications and uses of the numeration systems are briefly indicated. These include the ranking of permutations, of permutations with repetitions and of Cayley-permutations; polyphase sorting and merging of large data files; irregularities of distribution of sequences, the Chinese Remainder Theorem, various games and a class of binary search trees called red trees.

It should be pointed out that not all the known numeration systems can be derived from Theorem 1. An example is the combinatorial numeration system

\[
N = \binom{a_n}{n} + \binom{a_{n-1}}{n-1} + \cdots + \binom{a_2}{2} + \binom{a_1}{1} \quad (0 \leq a_1 < a_2 < \cdots < a_n)
\]

(see e.g. Lehmer [12]). There is a way of generalizing Theorem 1 (from numbers to infinite sets) so as to include also the combinatorial representation, but we prefer at this stage to keep our results as simple as possible.

To keep the discussion simple, we state and prove our results for nonnegative integers only. At the end we indicate the slight modifications necessary to extend the results to any integer.

2. Two Basic Numeration Systems

Let \(1 = u_0 < u_1 < u_2 < \ldots \) be a finite or infinite sequence of integers. Let \(N\) be any nonnegative integer, and suppose that \(u_n\) is the largest number in the sequence not exceeding \(N\) (except that we let \(n = 0\) if \(N = 0\)). Dividing \(N\) by \(u_n\) and iterating gives

\[
N = d_n u_n + r_n, \quad 0 \leq r_n < u_n
\]

\[
r_n = d_{n-1} u_{n-1} + u_{n-1}, \quad 0 \leq r_{n-1} < u_{n-1}
\]

\[
r_{n-1} = d_{n-2} u_{n-2} + r_{n-2}, \quad 0 \leq r_{n-2} < u_{n-2}
\]

\[\vdots\]

\[
r_i = d_{i+1} u_i + r_i, \quad 0 \leq r_i < u_i
\]

\[\vdots\]

\[
r_2 = d_1 u_1 + r_1, \quad 0 \leq r_1 < u_1
\]

\[
r_1 = d_0 u_0
\]
Collecting terms we get
\[ N = n d u + u_{n-2} + \cdots + d_{0} u_{0} \quad (d_{i} \geq 0, \ i \geq 0). \]
This is the representation of \( N \) in the numeration system \( S = (u_{0}, u_{1}, u_{2}, \ldots) \). Any \( N \) represented in this form is also said to be representable by \( S \).
The above process shows that every nonnegative integer is representable by \( S \). Note that
\[ r_{i+1} = d_{i+1} u_{i+1} + d_{i} u_{i} + \cdots + d_{0} u_{0} \quad (i \geq 0). \]
We show that conversely, any number \( N = \sum_{i=0}^{n} d_{i} u_{i} \) satisfying
\[ d_{i} u_{i} + d_{i-1} u_{i-1} + \cdots + d_{0} u_{0} \quad (i \geq 0), \]
is the unique representation of \( N \) by \( S \). In fact,

**Theorem 1.** Let \( 1 = u_{0} < u_{1} < \cdots \) be any finite or infinite sequence of integers. Any nonnegative integer \( N \) is representable by \( S = (u_{0}, u_{1}, u_{2}, \ldots) \) in the form \( N = \sum_{i=0}^{n} d_{i} u_{i} \). This representation is unique if and only if (1) holds.

Proof. It remains only to establish uniqueness. Suppose that \( N \) has two representations:
\[ N = \sum_{i=0}^{n} c_{i} u_{i} = \sum_{i=0}^{n} d_{i} u_{i}, \]
where the digits \( c_{i} \) and \( d_{i} \) are nonnegative and satisfy (1). Let \( i \) be the largest integer such that \( c_{i} \neq d_{i} \), i.e., say \( c_{i} > d_{i+1} \). Then
\[ u_{i+1} = (c_{i} - d_{i+1}) u_{i} + \cdots + (d_{0} - c_{i}) u_{0}, \]
 contradicting (1).

Conversely suppose that (1) does not hold, that is,
\[ d_{i} u_{i} + d_{i-1} u_{i-1} + \cdots + d_{0} u_{0} < u_{i+1}, \]
for some \( i \geq 0 \). Let \( N = d_{i} u_{i} + d_{i-1} u_{i-1} + \cdots + d_{0} u_{0} \), and let \( u_{n} \) be the largest number in \( S \) not exceeding \( N \). Then \( n < i+1 \). As was shown by the sequence of divisions preceding (1), there is a representation of the form \( N = \sum_{i=0}^{n} c_{i} u_{i} \) with \( c_{n} \neq 0 \). Thus \( N \) has two distinct representations.

The existence of the representation has been shown in Yaglom and Yaglom (Ch. 8), where also the sufficiency of the uniqueness is stated.

Incidentally, note that (1) implies
\[ 0 \leq d_{i} < \frac{i+1}{u_{i}} \quad (i \geq 0). \]

Now sometimes (2) implies (1) and sometimes it does not. When the \( u_{i} \) are defined recursively, the situation depends on the length of the recurrence relation! If the recurrence relation contains only one term \( (u_{n} = b_{n} u_{n-1}) \), then (2) does imply (1). Therefore for the more conventional numeration systems such as decimal, binary, mixed radix and factorial systems, (2) is a necessary and sufficient condition for uniqueness. For systems in which the recurrence relation contains more than one term, (2) is only a necessary but not a sufficient condition. This will become clear from Theorem 2 below.

For \( m > 1 \), let \( b_{1} = b_{1}^{(n)}, b_{2}, \ldots, b_{m} \) be integers satisfying
\[ 1 < b_{i} < b_{i+1}^{(n)} \quad (i \geq 1), \]
for all \( n > 0 \). Note that \( b_{i}, \ldots, b_{m} \) are constants, but \( b_{1}^{(n)} \) may depend on \( n \). Suppose that
\[ u_{m+1} < u_{m+2} < \cdots < u_{n-1} \]
are fixed nonnegative integers, \( u_{0}, \ldots, u_{m} \).

If \( m = 1 \), we have by (2), \( d_{1} < \frac{i+1}{u_{i}} \). If \( m > 1 \), then
\[ d_{1} < (b_{1}^{(n)} + b_{2} u_{1} + \cdots + b_{m} u_{m})^{(n)}/u_{i}, \]
which is equivalent to (1) with \( m = 1 \). Thus
\[ 0 < d_{1} < 1 \quad (i \geq 1). \]

When is an integer uniquely representable by the system \( (u_{0}, u_{1}, \ldots) \) thus defined? Here is the answer.

**Theorem 2.** Let \( S = (u_{i}) \) be a sequence of the form (3). Any nonnegative integer can be expressed in the form \( N = \sum_{i=0}^{n} d_{i} u_{i} \), where the digits \( d_{i} \) satisfy (4). The representation is unique over \( S \) if and only if the following two-fold condition holds:

(i) For any \( j \) satisfying \( 1 \leq j < m-1 \), if
\[ (d_{k}^{(k)} d_{k-1}^{(k)} \ldots d_{k-j}^{(k)} b_{1}^{(k)} b_{2}^{(k)} \ldots b_{j}) \]
then \( d_{k-j} < b_{j+1} \) and if (5) holds with \( j = m-1 \), then \( d_{k-m} < b_{m} \).

(ii) If (5) holds for any \( j \) satisfying \( 1 \leq j \leq k-1 \), then \( d_{k-j} < b_{j+1} \), and if (5) holds with \( j = k \), then \( d_{k-m} < b_{m} \).

We point out that subconditions (i) and (ii) are both concerned with blocks of consecutive digits. They differ only in the location of these blocks: in (i) the right-hand digit of a block of maximal length \( m \) coincides with \( d_{j} \) for some \( j \geq 0 \); whereas in (ii), the right-hand digit of a block of smaller size \( k+1 < m \) already coincides with \( d_{j} \).

Further note that if \( m = 1 \), then subcondition (i) merely restates \( 0 < d_{1} < b_{1}^{(m)} \) (\( k = 1 \)), which is part of (4), and (ii) is empty. We also remark that Theorem 2 does not consider the most general case.
(for example, some negative coefficients could be permitted in the recurrence relation), but it suffices for deriving in a simple manner all but one of the numeration systems of interest to us.

Proof. The existence of the representation \( N = \sum_{i=0}^{m} d_i u_i \) follows as in the proof of Theorem 1 with the digit bounds of (4). For proving uniqueness, assume that (i) does not hold. Suppose first that there is some \( j \) satisfying \( 1 \leq j \leq m - 2 \) for which (5) holds but \( d_{k-j} > b_{k+j} \). Then

\[
N = \sum_{i=0}^{k-1} d_i u_i + d_{k-1} u_{k-1} + \sum_{i=k+1}^{j-1} d_i u_i + d_j u_j + \sum_{i=j+1}^{m} d_i u_i.
\]

violating (1). Hence the representation is not unique by Theorem 1. Secondly suppose that (5) holds with \( j = m - 1 \) but \( d_{m-1} > b_m \). Then

\[
N = \sum_{i=0}^{m-1} d_i u_i + d_{m-1} u_{m-1} + d_m u_m + \sum_{i=m+1}^{k-1} d_i u_i + d_{k-1} u_{k-1} + \sum_{i=k+1}^{j-1} d_i u_i + d_j u_j + \sum_{i=j+1}^{m} d_i u_i.
\]

again violating (1). If we assume that (ii) does not hold, then the same arguments show again that (1) is violated. So suppose that the condition holds. Write \( n + 1 = g m + r \), \( 0 \leq r < m \). Then

\[
N \leq b_{n+1} u_0 + b_{n+2} u_1 + \ldots + b_{n+m} u_m + b_{n+2} u_{n+2} + b_{n+3} u_{n+3} + \ldots + b_{n+m} u_{n+m} + b_{n+2} u_{n+2} + b_{n+3} u_{n+3} + \ldots + b_{n+m} u_{n+m}.
\]

where, by (i) and (ii), \( d_0 = b - 1 \) if \( r = 0 \); \( d_0 = \sum_{i=0}^{m-r} b_i u_i - 1 \) (if \( r > 0 \)).

Adding

\[
0 = (u_{n+1} - u_{n+1} - m) + (u_{n+1} - m - u_{n+1} - 2m) + \ldots \]

\[
+ (u_{n+1} - m - 1) + (u_{n+1} - m - 2) + \ldots \]

so condition (1) is satisfied. The result now follows from Theorem 1.

3. A Spectrum of Numeral Systems

We shall now use Theorem 2 to derive several families of useful numeral systems. Existence of these families is evident by the procedure just preceding Theorem 1. It will therefore suffice to demonstrate the digit bounds and uniqueness. Recall that \( u_0 = 1 \) for all systems.

Polynomial systems. Let \( b > 1 \) be a fixed integer and let \( u_n = b^n \), that is, \( u_{n+1} = bu_n \) \( (n \geq 0) \). Let \( N \) be any nonnegative integer. Since the recurrence for the \( u_i \) has length \( 1 \), Theorem 2 implies that the representation \( N = \sum_{i=0}^{n} d_i u_i \) is unique if and only if \( d_i \leq b \) \( (i \geq 0) \). This gives the most common used numeral systems, such as the decimal \( (b = 10) \) and the binary \( (b = 2) \) system.

Mixed radix. Let \( i = a_i a_{i-1} a_{i-2} \ldots \) be any sequence of integers with \( a_i \geq 1 \) \( (i \geq 1) \), and let \( u_n = a_i a_{i-1} a_{i-2} \ldots u_0 \) that is, \( u_{n+1} = a_i u_n \) \( (n \geq 0) \). By the above argument, the representation \( N = \sum_{i=0}^{m} d_i a_i \) is unique if and only if \( d_i \leq a_i \) \( (i \geq 0) \). The mixed radix representation has been used for a constructive proof of the generalized Chinese Remainder Theorem (see Braikenl, Knuth) \( \text{[Sect. 4.1.2]} \); and in conjunction with other numeral systems, for ranking permutations with repetitions and Cayleypermutations. The latter method has been applied for compressing and partitioning large dictionaries in order to enable the storage of their "information bearing" parts in high-speed memory.

Factorial representation. This is the special case of the mixed radix representation where \( a_i = i+1 \) \( (n \geq 0) \). Thus the representation \( N = \sum_{i=m}^{n} d_i i! \) is unique if and only if \( d_i \leq i \) \( (i \geq 1) \). The factorial representation has been used for ranking permutations; see Lehmer and Even \( \text{[Ch. 1]} \).

Reflected factorial representation. To represent a nonnegative integer \( N \), select \( b \) with \( b > N \), and let \( u_n = h/(n-1) \), that is, \( u_{n+1} = u_n \) \( (n \geq 0) \).

Since again the recurrence has length \( 1 \) only, the representation \( N = \sum_{i=0}^{h-2} d_i i! \) is unique if and only if \( d_i \leq h \) \( (0 \leq i \leq h-2) \). The reflected factorial representation has also been used for ranking permutations \( \text{[Ch. 1]} \).

Up to this point all systems used only a one-term recurrence relation for the \( u_i \) (the case \( m = 1 \) in Theorem 2). This produced the better known numeral systems. The more exotic systems are obtained for \( m > 1 \). In these cases requirement (4) does not suffice to insure uniqueness, and the condition of Theorem 2 is needed to guarantee it. We start with
an example illustrating the case \( m = 2 \).

Continued fraction representation. Let \( a \) be an irrational number satisfying \( 1 < a < 2 \). Then \( a \) has a unique simple continued fraction expansion of the form

\[
a = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}
\]

where the \( a_i \) are positive integers. Its convergents \( p_n/q_n = (1, a_1, a_2, \ldots, a_n) \) satisfy the recursion

\[
P_{n+1} = a_n P_n + P_{n-1}, \quad Q_{n+1} = a_n Q_n + Q_{n-1}, \quad n \geq 1
\]

See e.g. Hardy and Wright (Ch. 10), Olds' or Perron. We prove,

**Theorem 3.** Every nonnegative integer can be represented uniquely in the form

\[
N = \sum_{i=0}^{k} t_i q_i, \quad \text{if } t_i \in \{0, 1, 2\}, \quad q_i \in \{1, 2\}
\]

and also in the form

\[
N = \sum_{i=0}^{k} t_i q_i, \quad \text{if } t_i \in \{0, 1, 2\}, \quad q_i \in \{1, 2\}
\]

Proof. Let \( u_i = p_i/q_i \) or \( q_i \) \((i \geq 1)\). The requirements (4) imply the bounds on the digits \( a_i \) and \( t_i \); and the condition of Theorem 2 implies that one of these digits attains its maximal value if the corresponding neighbor must vanish.

If \( a_1 = 1 \) \((i \geq 1)\), then \( a = [1] = (1+\sqrt{5})/2 \) is the golden ratio (where \( \phi \) denotes the infinite concatenation of \( a \) with itself). In this case the system (6) becomes the Fibonacci numeration system which is a binary system (digits 0 and 1 only), with the proviso that two adjacent 1’s never occur. See Zeckendorf. This system lies behind the Fibonacci search (see Knuth §6.2.1); it has also been used by Welsh and Knuth for giving the strategy of a game on a pile of tokens.

Numeration systems of the form (6) and (7) can also be defined for rational \( a \). An interesting relationship exists between \( n \) expressed in the system (7) and \([na]\) and \([n\hat{a}]\) in the system (6), where

\[
a^{-1} + \hat{a}^{-1} = 1.
\]

This relationship is particularly interesting for the special case \( a = [1, a] \) where \( a \) is any positive integer. It can be utilized for giving a winning strategy for generalized Wythoff games both in normal play and in misère play. The class of zero trees consolidates the winning strategies of those games. The case \( a = 1 \) gives a strategy for the classical Wythoff game.

We consider next an example of an arbitrary length recurrence relation.

The \( m \)th order Fibonacci system. The \( m \)th order Fibonacci numbers \( (m \geq 1) \) are defined by

\[
u_{m+1} = u_{m+2} = \ldots = u_2 = 0, \quad u_{m-1} = u_0 = 1.
\]

This definition gives the ordinary Fibonacci numbers for \( m = 2 \).

It follows directly from Theorem 2 that the binary system

\[
N = \frac{1}{\sum_{i=0}^{\infty} t_i u_i, \quad (0 \leq t_i \leq 1, \quad 0 \leq i \leq n)}
\]

is a unique numeration system if and only if it contains no run of \( m \) consecutive 1’s. Since such a system exists for every \( m \geq 2 \), there are infinitely many binary systems as claimed at the beginning of the paper.

4. Another Continued Fraction System

Let \( a = [1, a_1, a_2, a_3, \ldots] \), that is \( a_{2n} = a, \) where \( a \) is any positive integer. Further, let \( u_n \) stand for either \( u_i \) or \( q_i \) the understanding being that in each formula involving \( u_i \), either all \( u_i \) for \( P_i \) or all stand for \( q_i \). We shall develop two numeration systems based on the numerators and denominators of the even convergents of \( a \). Let us start with two auxiliary results on the even convergents. Throughout we let

\[
\epsilon = \begin{cases} 0 & \text{if } a_{2n+1} = a_{2n+2} \text{ for all } \geq n \geq 1, \\ 1 & \text{if } a_{2n+1} = a_{2n+2} \text{ for all } n \geq 1. \\
\end{cases}
\]

**Lemma 1.** The even convergents of \( a \) satisfy

\[
u_{2n+1} = a - u_n, \quad u_0 = 1, \quad u_{2n} = (a_{2n+1} + 2)u_{2n-2} - u_{2n-1} - u_{2n-2} - u_{2n-4} \quad (n \geq 1).
\]

**Proof.** For \( n \geq 1 \) we have,

\[
u_{2n+1} = a - u_{2n+1} = (a_{2n+1} + 2)u_{2n-2} - u_{2n-1} - u_{2n-2} - u_{2n-4} \quad (n \geq 1).
\]

**Lemma 2.** Let \( 0 < k < \epsilon \). Then

\[
u_{2k+2} = a_{2k}u_{2k+1} + a_{2k}u_{2k} + 2a_{2k+1}u_{2k+2} + a_{2k+2}u_{2k+3} + \ldots
\]

**Proof.** By Lemma 1,

\[
u_{2k+2} = a_{2k+1}u_{2k+1} + a_{2k+1}u_{2k} + 2a_{2k+1}u_{2k+2} + a_{2k+2}u_{2k+3} + \ldots
\]

We consider next an example of an arbitrary length recurrence relation.
We are now ready to present our last family of numeration systems.

Theorem 4. Every nonnegative integer can be represented in the form $N = \sum_{i=0}^{n} d_i 2^i$, where the digits $d_i$ satisfy

$$0 \leq d_i \leq a_{2i+1} + 1 \quad (i \geq 0), \quad 0 \leq d_0 \leq a(1+e). \quad (8)$$

The representation of $N$ is unique if and only if only the following condition holds: If for some $0 \leq k \leq n$, $d_k$ and $d_{k+1}$ attain their maximal values, then there exists $j$ satisfying $k < j \leq l$ (so actually $k < j \leq l$) such that $d_j < a_{2j+1}$.

Proof. The existence of the representation follows again by the algorithm just prior to Theorem 1. That method (see (2)) requires

$$d_{2i} = \frac{a_{2i+1} + 2}{u_{2i}} \cdot u_{2i+1}, \quad (i \geq 0),$$

which implies the bounds (8). For proving uniqueness assume first that the condition does not hold. If $k > 0$, then

$$\Sigma_{i=0}^{k} d_{2i} u_{2i} > \Sigma_{i=0}^{k} a(a_{2i+1} u_{2i} + \ldots + a_{2k+1} u_{2k})$$

$$+ u_{2k} + u_{2k} = u_{2k+2} + u_{2k-2} > u_{2k+2}$$

by Lemma 2. Since (1) is violated, Theorem 1 implies that the representation is not unique. If $k = 0$, then

$$\Sigma_{i=0}^{k} d_{2i} u_{2i} > a(a_{2i+1} u_{2i} + \ldots + a_{2k+1} u_{2k})$$

$$+ u_{2k} + u_{2k} = u_{2k+2} + u_{2k} + (\epsilon a-1) u_0 = u_{2k+2}$$

again violating (1). Now suppose that the condition

$$N = \Sigma_{i=0}^{n} d_{2i} u_{2i}$$

is fulfilled. Then $N = \Sigma_{i=0}^{n} d_{2i} u_{2i}$ is maximal for $i = n$, that is, $d_{2n} = a_{2n+1} + 1$. Now $d_{2k}$ maximal implies $d_{2j} < a_{2j+1}$ for some $k < j \leq l$. Since

$$d_{2i} u_{2i} > (i \geq 0), \quad N \text{ increases if we let } d_{2j} = a_{2j+1} \text{ for all } k < j \leq l, \quad \text{put } k = 0 \text{ and decrease the maximal value of } d_{2k} \text{ by 1. Thus}$$

$$N \leq a_{2n+1} u_{2n} + a_{2n+1} u_{2n-2} + \ldots + a_{2k} u_{2k} + a_{2k-1} u_{2k-2} \ldots + a_{2k+1} u_{2k+1} + 1$$

$$+ u_{2k} + e u_{2k-1} = u_{2n+2} + u_2 + (\epsilon a-1) u_0 = u_{2n+2} + 1$$

by Lemma 2. Thus condition (1) is satisfied and so the result follows by Theorem 1.

In the "Fibonacci case", that is the special case where $\alpha = \{1\}$, the $p_i$-system of Theorem 4 becomes a rather curious ternary numeration system since (8) now implies $0 \leq d_{2i} \leq 2 \quad (i \geq 0).$ In this case the condition of Theorem 4 states that between any two digits 2 there must be a digit 0. This special case was used by Chung and Graham to investigate irregularities of distribution of sequences.

We finally remark that for representing a nonnegative integer $N$ in any of the above numeration systems, represent $|N|$ and then reverse the signs of all the digits. In general for representing any $N$, the digits are either all nonnegative or all nonpositive. The changes needed in the proofs are essentially to replace conditions on digits by the same conditions on their absolute values. Specifically, (1) and (2) have to be replaced by

$$|d_{2i} + \ldots + d_{2i+1}| < u_{i+1} \quad (i \geq 0) \quad \text{and} \quad |d_i| < u_{i+1}/u_i \quad (i \geq 0)$$

respectively, (4) by

$$|d_2| < b_{i+1} \quad (i \geq 1), \quad |d_0| < b_1 + \Sigma_{j=2}^{m} b_j \quad \text{if } m > 1$$

$$|d_1| < b_{i+1} \quad \text{if } m = 1,$$

and (5) by

$$|d_{k-1}|, |d_{k-1}|, \ldots, |d_{k-1}| = (b_{k+1}, b_2, \ldots, b_1).$$

In Theorem 2, the two inequalities on the digits in (1) become $|d_{k-1}| \leq b_{j+1}$ and $|d_{k-1}| > b_i$; those of (ii) become $|d_{k-1}| \leq b_{j+1}$ and $|d_0| \leq \Sigma_{i=0}^{k+1} b_{i+1}$.

In (6), two of the conditions are replaced by $|s_i| \leq a_{i+1}$ and $|s_{i+1}| = a_{i+2}$ and in (7) three conditions are replaced by $|t_j| < a_{j+1}, \quad |t_j| < a_{j+1}$ and $|t_{j+1}| = a_{j+1}$. Finally (8) becomes

$$|d_{2i}| < a_{2i+1} + 1 \quad (i \geq 1), \quad |d_0| < a(1+e).$$

References

6. A.S. Fraenkel, J. Levitt and M. Shimshoni, Characterization of the set of values \( f(n) = [na] \), \( n = 1, 2, \ldots \), Discrete Math. 2 (1972) 335-345.


