SYSTEMS OF NUMERATION

Aviezri S. Fraenkel

Department of Applied Mathematics
Weizmann Institute of Science, Rehovot 76100, Israel

Abstract

A numeration system is a set of integers (basis elements) such that every integer can be represented uniquely over the set using integer digits of bounded size. Such systems are scattered in many fields in mathematics and computer science. Many of the known ones and new ones are unified and derived from a basic result on recursively defined basis elements. Applications are indicated.

1. Introduction

There are many ways of representing an integer uniquely! The best known method is the decimal system. Whereas the Maya Indians used base 20 (using the fingers on hands and feet), some of the human race became recently more primitive using the binary system instead, being influenced by the computer race which, for electronic reasons, is zealously addicted to the binary system. It may be of interest to computers to know that there are actually infinitely many binary systems!

Somewhat less known systems of numeration include mixed radix, factorial representation, and exotic systems based on recurrence relations, a special case of which is the Fibonacci system of numeration. So there are many ways of representing an integer uniquely; many ways, that is, in each of which an integer can be represented uniquely.

These and other systems of numeration normally hide in various unexpected places, where they are applied for varied purposes. Typically, when the need for a numeration system arises, it is defined and an ad hoc proof of its capability to represent integers uniquely is given. The purpose of this article is to unify these results and show how they can be derived simply and uniformly.

A very simple yet general system of numeration is presented in Theorem 1. It may be used to derive all the numeration systems we intend to present, but some repetitive argumentation is involved. We prefer instead to use Theorem 1 to derive a general numeration system based on recursively defined basis elements. This is done in Theorem 2, which sheds more light on the nature of numeration systems than Theorem 1. Our numeration systems are then derived from Theorem 2. All but one. The exceptional system is based on a recurrence relation with a negative coefficient, whereas the recurrence relations of Theorem 2 contain only positive coefficients. The exceptional system is therefore derived directly from Theorem 1.

Theorems 1 and 2 are given in Section 2. The derivation of the numeration systems from Theorem 2 is carried out in Section 3 in a rather slicker way than Theorem 1 would permit. The exceptional system is derived in the final Section 4. Applications and uses of the numeration systems are briefly indicated. These include the ranking of permutations, of permutations with repetitions and of Cayley-permutations; polyphase sorting and merging of large data files, irregularities of distribution of sequences, the Chinese Remainder Theorem, various games and a class of binary search trees called cedar trees.

It should be pointed out that not all the known numeration systems can be derived from Theorem 1. An example is the combinatorial numeration system

$$N = \binom{a_n}{n} + \binom{a_{n-1}}{n-1} + \cdots + \binom{a_2}{2} + \binom{a_1}{1} \quad (0 \le a_1 < a_2 < \cdots < a_n)$$

(see e.g. Lehmer[12]). There is a way of generalizing Theorem 1 (from numbers to infinite sets) so as to include also the combinatorial representation, but we prefer at this stage to keep our results as simple as possible.

To keep the discussion simple, we state and prove our results for nonnegative integers only. At the end we indicate the slight modifications necessary to extend the results to any integer.

2. Two Basic Numeration Systems

Let $1 = u_0 < u_1 < u_2 < \ldots$ be a finite or infinite sequence of integers. Let N be any nonnegative integer, and suppose that u_n is the largest number in the sequence not exceeding N (except that we let $n=0$ if $N=0$). Dividing N by u_n and iterating gives

$$\begin{aligned}
N &= d_n u_n + r_n, & 0 &\le r_n < u_n\\
r_n &= d_{n-1}u_{n-1} + r_{n-1}, & 0 &\le r_{n-1} < u_{n-1}\\
r_{n-1} &= d_{n-2}u_{n-2} + r_{n-2}, & 0 &\le r_{n-2} < u_{n-2}\\
&\vdots & &\vdots\\
r_{i+1} &= d_i u_i + r_i, & 0 &\le r_i < u_i\\
&\vdots & &\vdots\\
r_2 &= d_1 u_1 + r_1, & 0 &\le r_1 < u_1\\
r_1 &= d_0 u_0 .
\end{aligned}$$

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Collecting terms we get

$$N = d_n u_n + d_{n-1}u_{n-1} + \cdots + d_0 u_0 \quad (d_i \geq 0,\ i \geq 0).$$

This is the representation of N in the numeration system $S = \{u_0, u_1, u_2, \ldots\}$. Any N represented in this form is also said to be representable by S. The above process shows that every nonnegative integer is representable by S. Note that

$$r_{i+1} = d_i u_i + d_{i-1}u_{i-1} + \cdots + d_0 u_0 < u_{i+1} \quad (i \geq 0).$$

We show that conversely, any number $N = \sum_{i=0}^{n} d_i u_i$ satisfying

$$d_i u_i + d_{i-1}u_{i-1} + \cdots + d_0 u_0 < u_{i+1} \quad (i \geq 0), \tag{1}$$

is the unique representation of N by S. In fact,

Theorem 1. Let $1 = u_0 < u_1 < u_2 < \ldots$ be any finite or infinite sequence of integers. Any nonnegative integer N is representable by $S = \{u_0, u_1, u_2, \ldots\}$ in the form $N = \sum_{i=0}^{n} d_i u_i$. This representation is unique if and only if (1) holds.

Proof. It remains only to establish uniqueness. Suppose that N has two representations:

$$N = c_n u_n + \cdots + c_0 u_0 = d_n u_n + \cdots + d_0 u_0,$$

where the digits c_i and d_i are nonnegative and satisfy (1). Let i be the largest integer such that $c_{i+1} \neq d_{i+1}$, say $c_{i+1} > d_{i+1}$. Then

$$u_{i+1} \leq (c_{i+1} - d_{i+1})u_{i+1} = (d_i - c_i)u_i + \cdots + (d_0 - c_0)u_0 \leq d_i u_i + \cdots + d_0 u_0,$$

contradicting (1).

Conversely suppose that (1) does not hold, that is,

$$d_i u_i + d_{i-1}u_{i-1} + \cdots + d_0 u_0 \geq u_{i+1}$$

for some $i \geq 0$. Let $N = d_i u_i + d_{i-1}u_{i-1} + \cdots + d_0 u_0$, and let u_n be the largest number in S not exceeding N. Then $n \geq i+1$. As was shown by the sequence of divisions preceding (1), there is a representation of the form $N = \sum_{j=0}^{n} c_j u_j$ with $c_n \neq 0$. Thus N has two distinct representations. ■

The existence of the representation has been shown in Yaglom and Yaglom[18] [Ch. 8], where also the sufficiency part of the uniqueness is stated.

Incidentally, note that (1) implies

$$0 \leq d_i < \frac{u_{i+1}}{u_i} \quad (i \geq 0). \tag{2}$$

Now sometimes (2) implies (1) and sometimes it does not. When the u_i are defined recursively, the situation depends on the length of the recurrence relation! If the recurrence relation contains only one term $(u_n = b^{(n)}u_{n-1})$, then (2) does imply (1). Therefore for the more conventional numeration systems such as decimal, binary, mixed radix and factorial systems, (2) is a necessary and sufficient condition for uniqueness. But for systems in which the recurrence relation contains more than one term, (2) is only a necessary but not a sufficient condition. This will become clear from Theorem 2 below.

For $m \geq 1$, let $b_1 = b_1^{(n)}, b_2, \ldots, b_m$ be integers satisfying

$$1 \leq b_m \leq \cdots \leq b_2 \leq b_1^{(n)}$$

for all $n \geq 1$. Note that $b_2, \ldots, b_m$ are constants, but $b_1 = b_1^{(n)}$ may depend on n. Suppose that

$$\left.\begin{array}{l} u_{-m+1}, u_{-m+2}, \ldots, u_{-1} \text{ are fixed nonnegative integers,} \\ u_0 = 1,\ u_n = b_1^{(n)}u_{n-1} + b_2 u_{n-2} + \cdots + b_m u_{n-m} \quad (n \geq 1). \end{array}\right\} \tag{3}$$

If $m = 1$, we have by (2), $d_i < u_{i+1}/u_i = b_1^{(i+1)}$. If $m > 1$, then

$$\begin{aligned} d_i &< (b_1^{(i+1)}u_i + b_2 u_{i-1} + \cdots + b_m u_{i-m+1})/u_i \\ &\leq b_1^{(i+1)} + (b_1^{(i)}u_{i-1} + b_2 u_{i-2} + \cdots + b_m u_{i-m})/u_i \\ &= b_1^{(i+1)} + 1 \quad (i \geq 1). \end{aligned}$$

Also $d_0 < u_1 = \sum_{j=1}^{m} b_j u_{1-j}$. Thus

$$\left.\begin{array}{l} 0 \leq d_i \leq b_1^{(i+1)} \ (i \geq 1),\ 0 \leq d_0 < b_1^{(1)} + \sum_{j=2}^{m} b_j u_{1-j} \quad (m > 1) \\ \qquad 0 \leq d_i < b_1^{(i+1)} \ (i \geq 0) \quad (m = 1). \end{array}\right\} \tag{4}$$

When is an integer uniquely representable by the system $\{u_0, u_1, \ldots\}$ thus defined? Here is the answer.

Theorem 2. Let $S = \{u_i\}$ be a sequence of the form (3). Any nonnegative integer can be expressed in the form $N = \sum_{i=0}^{n} d_i u_i$, where the digits d_i satisfy (4). The representation is unique over S if and only if the following two-fold condition holds:

(i) For any j satisfying $1 \leq j \leq m-2$, if

$$(d_k, d_{k-1}, \ldots, d_{k-j+1}) = (b_1^{(k+1)}, b_2, \ldots, b_j), \tag{5}$$

then $d_{k-j} \leq b_{j+1}$; and if (5) holds with $j = m-1$, then $d_{k-m+1} < b_m$ $(k \geq m-1)$.

(ii) If (5) holds for any j satisfying $1 \leq j \leq k-1$, then $d_{k-j} \leq b_{j+1}$; and if (5) holds with $j = k$, then $d_0 < \sum_{i=k+1}^{m} b_i u_{k+1-i}$ $(1 \leq k < m-1)$.

We point out that subconditions (i) and (ii) are both concerned with blocks of consecutive digits. They differ only in the location of these blocks: in (i) the right-hand digit of a blcok of maximal length m coincides with d_j for some $j \geq 0$; whereas in (ii), the right-hand digit of a block of smaller size $k+1 < m$ already coincides with d_0.

Further note that if $m = 1$, then subcondition (i) merely restates $0 \leq d_k < b_1^{(k+1)}$ $(k \geq 0)$ which is part of (4), and (ii) is empty. We also remark that Theorem 2 does not consider the most general case (for example, some negative coefficients could be permitted in the recurrence relation), but it suffices for deriving in a simple manner all but one of the numeration systems of interest to us.

Proof. The existence of the representation $N=\sum_{i=0}^{n} d_i u_i$ follows as in the proof of Theorem 1 with the digit bounds of (4). For proving uniqueness, assume that (i) does not hold. Suppose first that there is some j satisfying $1 \le j \le m-2$ for which (5) holds but $d_{k-j} > b_{j+1}$. Then

$$\sum_{i=0}^{k} d_i u_i \ge \sum_{i=k-j}^{k} d_i u_i \ge b_1^{(k+1)} u_k + \sum_{i=2}^{j+1} b_i u_{k+1-i} + u_{k-j}$$

$$= b_1^{(k+1)} u_k + \sum_{i=2}^{j+1} b_i u_{k+1-i} + b_1^{(k-j)} u_{k-j-1} + \sum_{i=2}^{m} b_i u_{k-j-i}$$

$$\ge b_1^{(k+1)} u_k + \sum_{i=2}^{m} b_i u_{k+1-i} = u_{k+1},$$

violating (1). Hence the representation is not unique by Theorem 1. Secondly suppose that (5) holds with $j=m-1$ but $d_{k-m+1} \ge b_m$. Then

$$\sum_{i=0}^{k} d_i u_i \ge \sum_{i=k+1-m}^{k} d_i u_i \ge b_1^{(k+1)} u_k + \sum_{i=2}^{m} b_i u_{k+1-i} = u_{k+1},$$

again violating (1). If we assume that (ii) does not hold, then the same arguments show again that (1) is violated. So suppose that the condition holds. Write $n+1 = qm + r$, $0 \le r < m$. Then

$$\begin{aligned} N \le\ & b_1^{(n+1)} u_n + b_2 u_{n-1} + \cdots + b_{m-1} u_{n+2-m} + (b_m - 1) u_{n+1-m} \\ &+ b_1^{(n+1-m)} u_{n-m} + b_2 u_{n-1-m} + \cdots + b_{m-1} u_{n+2-2m} \\ &\qquad + (b_m - 1) u_{n+1-2m} \\ &+ \cdots \\ &+ b_1^{(n+1-(q-1)m)} u_{n-(q-1)m} + b_2 u_{n-1-(q-1)m} + \cdots \\ &\qquad + b_{m-1} u_{n+2-qm} + (b_m - 1) u_{n+1-qm} \\ &+ b_1^{(n+1-qm)} u_{n-qm} + b_2 u_{n-1-qm} + \cdots + b_{r-1} u_{n+2-r-qm} \\ &\qquad + d_0 u_0, \end{aligned}$$

where, by (i) and (ii), $d_0 = b_m - 1$ (if $r=0$); $d_0 = \sum_{i=0}^{m-r} b_{r+i} u_{-i} - 1$ (if $r>0$).

Adding

$$0 = (u_{n+1-m} - u_{n+1-m}) + (u_{n+1-2m} - u_{n+1-2m}) + \cdots + (u_{n+1-qm} - u_{n+1-qm}) + (u_0 - u_0)$$

to the right-hand-side, we get

$$N \le (u_{n+1} - u_{n+1-m}) + (u_{n+1-m} - u_{n+1-2m}) + \cdots + (u_{n+1-(q-1)m} - u_{n+1-qm}) + (u_{n+1-qm} - 1) = u_{n+1} - 1,$$

so condition (1) is satisfied. The result now follows from Theorem 1. ■

3. A Spectrum of Numeration Systems

We shall now use Theorem 2 to derive several families of useful numeration systems. Existence of these families is evident by the procedure just preceding Theorem 1. It will therefore suffice to demonstrate the digit bounds and uniqueness. Recall that $u_0 = 1$ for all systems.

Polynomial systems. Let $b>1$ be a fixed integer and let $u_n = b^n$, that is, $u_{n+1} = b u_n$ $(n \ge 0)$. Let N be any nonnegative integer. Since the recurrence for the u_i has length 1, Theorem 2 implies that the representation $N = \sum_{i=0}^{n} d_i b^i$ is unique if and only if $0 \le d_i < b$ $(i \ge 0)$. This gives the most commonly used numeration systems, such as the decimal $(b=10)$ and the binary $(b=2)$ system.

Mixed radix. Let $1 = a_0, a_1, a_2, \ldots$ be any sequence of integers with $a_i > 1$ $(i \ge 1)$, and let $u_n = a_0 a_1 \ldots a_n$, that is, $u_{n+1} = a_{n+1} u_n$ $(n \ge 0)$. By the above argument, the representation $N = \sum_{i=0}^{n} d_i a_0 \ldots a_i$ is unique if and only if $0 \le d_i < a_{i+1}$ $(i \ge 0)$. The mixed radix representation has been used for a constructive proof of the generalized Chinese Remainder Theorem (see Fraenkel[3], Knuth[10] [Sect. 4.3.2]); and in conjunction with other numeration systems, for ranking permutations with repetitions and Cayley-permutations[14]. The latter method has been applied for compressing and partitioning large dictionaries in order to enable the storage of their "information bearing" parts in high-speed memory[7].

Factorial representation. This is the special case of the mixed radix representation where $a_n = n+1$ $(n \ge 0)$. Thus the representation $N = \sum_{i=1}^{n} d_i i!$ is unique if and only if $d_i \le i$ $(i \ge 1)$. The factorial representation has been used for ranking permutations; see Lehmer[12] and Even[2] [Ch. 1].

Reflected factorial representation. To represent a nonnegative integer N, select h with $h! > N$, and let $u_n = h!/(h-n)!$, that is, $u_{n+1} = u_n(h-n)$ $(n \ge 0)$. Since again the recurrence has length 1 only, the representation $N = \sum_{i=0}^{h-2} d_i h!/(h-i)!$ is unique if and only if $0 \le d_i < h-i$ $(0 \le i \le h-2)$. The reflected factorial representation has also been used for ranking permutations[2] [Ch. 1].

Up to this point all systems used only a one-term recurrence relation for the u_i (the case $m=1$ in Theorem 2). This produced the better known numeration systems. The more exotic systems are obtained for $m>1$. In these cases requirement (4) does not suffice to insure uniqueness, and the condition of Theorem 2 is needed to guarantee it. We start with an example illustrating the case $m = 2$.

Continued fraction representation. Let α be an irrational number satisfying $1 < \alpha < 2$. Then α has a unique simple continued fraction expansion of the form

$$\alpha = 1 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} = [1, a_1, a_2, a_3, \ldots],$$

where the a_i are positive integers. Its convergents $p_n/q_n = [1, a_1, a_2, \ldots, a_n]$ satisfy the recursion

$$p_{-1} = 1,\ p_0 = 1,\ p_n = a_n p_{n-1} + p_{n-2} \quad (n \geq 1)$$

$$q_{-1} = 0,\ q_0 = 1,\ q_n = a_n q_{n-1} + q_{n-2} \quad (n \geq 1).$$

See e.g. Hardy and Wright[8] [Ch. 10], Olds[15] or Perron[16]. We prove,

Theorem 3. Every nonnegative integer can be represented uniquely in the form

$$N = \sum_{i=0}^{k} s_i p_i,\ 0 \leq s_i \leq a_{i+1};\ s_{i+1} = a_{i+2} \Rightarrow s_i = 0 \quad (i \geq 0), \tag{6}$$

and also in the form

$$N = \sum_{i=0}^{\ell} t_i q_i,\ 0 \leq t_0 < a_1,\ 0 \leq t_i \leq a_{i+1};\ t_i = a_{i+1} \Rightarrow t_{i-1} = 0 \quad (i \geq 1). \tag{7}$$

Proof. Let $u_i = p_i$ or q_i $(i \geq -1)$. The requirements (4) imply the bounds on the digits s_i and t_i; and the condition of Theorem 2 implies that if one of these digits attains its maximal value then its right-hand-side neighbor must vanish. ∎

If $a_i = 1$ $(i \geq 1)$, then $\alpha = [\dot{1}] = (1+\sqrt{5})/2$ is the golden ratio (where $\dot{x}$ denotes the infinite concatenation of x with itself). In this case the system (6) becomes the Fibonacci numeration system which is a binary system (digits 0 and 1 only), with the proviso that two adjacent 1's never occur. See Zeckendorf[19]. This system lies behind the Fibonacci search (see Knuth[11] [Sect. 6.2.1]); it has also been used by Whinihan[17,9] [Sect. 1.2.8] for giving the strategy of a game on a pile of tokens.

Numeration systems of the form (6) and (7) can also be defined for rational α. An interesting relationship exists between n expressed in the system (7) and $\lfloor n\alpha \rfloor$ and $\lfloor n\beta \rfloor$ in the system (6), where $\alpha^{-1} + \beta^{-1} = 1$. This relationship is particularly interesting for the special case $\alpha = [1, \dot{a}]$ where a is any positive integer[5]. It can be utilized for giving a winning strategy for generalized Wythoff games both in normal play[4] and in misère play[5]. The class of cedar trees consolidates the winning strategies of those games. The case $a = 1$ gives a strategy for the classical Wythoff game[18].

We consider next an example of an arbitrary length recurrence relation.

mth order Fibonacci system. mth order Fibonacci numbers $(m \geq 2)$ are defined by

$$u_{-m+1} = u_{-m+2} = \ldots = u_{-2} = 0,\ u_{-1} = u_0 = 1,$$

$$u_n = u_{n-1} + u_{n-2} + \ldots + u_{n-m} \quad (n \geq 1).$$

This definition gives the ordinary Fibonacci numbers for $m = 2$.

It follows directly from Theorem 2 that the binary system

$$N = \sum_{i=0}^{n} d_i u_i \quad (0 \leq d_i \leq 1,\ 0 \leq i \leq n)$$

is a unique numeration system if and only if it contains no run of m consecutive 1's. Since such a system exists for every $m \geq 2$, there are infinitely many binary systems as claimed at the beginning of the paper.

mth order Fibonacci numbers have been used by Knuth[11] [Sect. 5.4.2] for polyphase merge of data runs stored on magnetic tape transports. The same numeration system but with different values for $u_{-m+1}, u_{-m+2}, \ldots, u_{-1}$ has been applied by Lynch[13] to polyphase sorting.

4. Another Continued Fraction System

Let $\alpha = [1, a_1, a, a_3, a, \ldots]$, that is $a_{2n} = a$, where a is any positive integer. Further, let u_n stand for either p_n or q_n, the understanding being that in each formula involving u_i, either all u_i stand for p_i or all stand for q_i. We shall develop two numeration systems based on the numerators and denominators of the even convergents of α. Let us start with two auxiliary results on the even convergents. Throughout we let

$$\varepsilon = \begin{cases} 0 & \text{if } u_i = q_i \\ 1 & \text{if } u_i = p_i \end{cases} \quad (i \geq -1).$$

Lemma 1. The even convergents of α satisfy

$$u_{-2} = 1 - \varepsilon a,\ u_0 = 1,\ u_{2n} = (aa_{2n-1} + 2)u_{2n-2} - u_{2n-4} \quad (n \geq 1).$$

Proof. For $n \geq 1$ we have,

$$\begin{aligned} u_{2n} &= au_{2n-1} + u_{2n-2} = a(a_{2n-1}u_{2n-2} + u_{2n-3}) + u_{2n-2} \\ &= aa_{2n-1}u_{2n-2} + (u_{2n-2} - u_{2n-4}) + u_{2n+2} \\ &= (aa_{2n-1} + 2)u_{2n-2} - u_{2n-4}. \end{aligned}$$ ∎

Lemma 2. Let $0 \leq k < \ell$. Then

$$u_{2\ell+2} = a(a_{2\ell+1}u_{2\ell} + a_{2\ell-1}u_{2\ell-2} + \ldots + a_{2k+3}u_{2k+2} + a_{2k+1}u_{2k}) + u_{2\ell} + u_{2k} - u_{2k-2}.$$

Proof. By Lemma 1,

$$u_{2\ell+2} = (aa_{2\ell+1} + 2)u_{2\ell} - u_{2\ell-2}$$

$$u_{2\ell} = (aa_{2\ell-1} + 2)u_{2\ell-2} - u_{2\ell-4}$$

$$u_{2\ell-2} = (aa_{2\ell-3} + 2)u_{2\ell-4} - u_{2\ell-6}$$

$\vdots$

$$u_{2k+6} = (aa_{2k+5} + 2)u_{2k+4} - u_{2k+2}$$
$$u_{2k+4} = (aa_{2k+3} + 2)u_{2k+2} - u_{2k}$$
$$u_{2k+2} = (aa_{2k+1} + 2)u_{2k} - u_{2k-2}\,.$$

Adding we get the claimed result. ■

We are now ready to present our last family of numeration systems.

Theorem 4. Every nonnegative integer can be represented in the form $N = \sum_{i=0}^{n} d_{2i}u_{2i}$, where the digits d_{2i} satisfy

$$0 \leq d_{2i} \leq aa_{2i+1} + 1 \ (i \geq 1), \quad 0 \leq d_0 \leq a(a_1 + \varepsilon)\,. \qquad (8)$$

The representation of N is unique if and only if the following condition holds: If for some $0 \leq k < \ell \leq n$, $d_{2\ell}$ and d_{2k} attain their maximal values, then there exists j satisfying $k < j < \ell$ (so actually $\ell - k \geq 2$) such that $d_{2j} < aa_{2j+1}$.

Proof. The existence of the representation follows again by the algorithm just prior to Theorem 1. That method (see (2)) requires

$$d_{2i} < \frac{u_{2i+2}}{u_{2i}} = \frac{(aa_{2i+1}+2)u_{2i} - u_{2i-2}}{u_{2i}} =$$
$$aa_{2i+1} + 2 - \frac{u_{2i-2}}{u_{2i}} \qquad (i \geq 0),$$

which implies the bounds (8). For proving uniqueness assume first that the condition does not hold. If $k > 0$, then

$$\sum_{i=0}^{\ell} d_{2i}u_{2i} \geq \sum_{i=k}^{\ell} d_{2i}u_{2i} \geq a(a_{2\ell+1}u_{2\ell} + \dots + a_{2k+1}u_{2k})$$
$$+ u_{2\ell} + u_{2k} = u_{2\ell+2} + u_{2k-2} > u_{2\ell+2}$$

by Lemma 2. Since (1) is violated, Theorem 1 implies that the representation is not unique. If $k = 0$, then

$$\sum_{i=0}^{\ell} d_{2i}u_{2i} \geq a(a_{2\ell+1}u_{2\ell} + \dots + a_3u_2 + a_1u_0)$$
$$+ u_{2\ell} + \varepsilon a u_0 = u_{2\ell+2} + u_{-2} + (\varepsilon a - 1)u_0 = u_{2\ell+2},$$

again violating (1). Now suppose that the condition is fulfilled. Then $N = \sum_{i=0}^{n} d_{2i}u_{2i}$ is maximal for $\ell = n$, that is, $d_{2n} = aa_{2n+1} + 1$. Now d_{2k} maximal implies $d_{2j} < aa_{2j+1}$ for some $k < j < \ell$. Since $u_{2i} \geq u_{2i-2}$ $(i \geq 0)$, N increases if we let $d_{2j} = aa_{2j+1}$ for all $k < j < \ell$, put $k = 0$ and decrease the maximal value of $d_{2k} = d_0$ by 1. Thus

$$N \leq a(a_{2n+1}u_{2n} + a_{2n-1}u_{2n-2} + \dots + a_3u_2 + a_1u_0)$$
$$+ u_{2n} + \varepsilon a u_0 - 1 = u_{2n+2} + u_{-2} + (\varepsilon a - 1)u_0 - 1 = u_{2n+2} - 1$$

by Lemma 2. Thus condition (1) is satisfied and so the result follows by Theorem 1. ■

In the "Fibonacci case", that is the special case where $\alpha = [\bar{1}]$, the p_i-system of Theorem 4 becomes a rather curious ternary numeration system since (8) now implies $0 \leq d_{2i} \leq 2$ $(i \geq 0)$. In this case the condition of Theorem 4 states that between any two digits 2 there must be a digit 0. This special case was used by Chung and Graham[1] to investigate irregularities of distribution of sequences.

We finally remark that for representing a negative integer N in any of the above numeration systems, represent $|N|$ and then reverse the signs of all the digits. In general for representing any N, the digits are either all nonnegative or all nonpositive. The changes needed in the proofs are essentially to replace conditions on digits by the same conditions on their absolute values. Specifically, (1) and (2) have to be replaced by

$$|d_iu_i + \dots + d_0u_0| < u_{i+1} \ (i \geq 0) \ \text{and} \ |d_i| < u_{i+1}/u_i \ (i \geq 0)$$

respectively, (4) by

$$|d_i| \leq b_1^{(i+1)} \ (i \geq 1), \ |d_0| < b_1^{(1)} + \sum_{j=2}^{m} b_j u_{1-j} \qquad (m > 1)$$
$$|d_i| < b_1^{(i+1)} \qquad (m = 1),$$

and (5) by

$$(|d_k|, |d_{k-1}|, \dots, |d_{k-j+1}|) = (b_1^{(k+1)}, b_2, \dots, b_j)\,.$$

In Theorem 2, the two inequalities on the digits in (i) become $|d_{k-j}| \leq b_{j+1}$ and $|d_{k-m+1}| < b_m$; those of (ii) become $|d_{k-j}| \leq b_{j+1}$ and $|d_0| < \sum_{i=k+1}^{m} b_i u_{k+1-i}$. In (6), two of the conditions are replaced by $|s_i| \leq a_{i+1}$ and $|s_{i+1}| = a_{i+2}$ and in (7) three conditions are replaced by $|t_0| < a_1$, $|t_i| \leq a_{i+1}$ and $|t_i| = a_{i+1}$. Finally (8) becomes

$$|d_{2i}| \leq aa_{2i+1} + 1 \ (i \geq 1), \quad |d_0| \leq a(a_1 + \varepsilon)\,.$$

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