

# CONDITIONS FOR THE DISTRIBUTIVITY OF MULTIPLICATION WITH RESPECT TO SET ADDITION AND THEIR EFFECT ON THE DESIGN OF ARRAY MULTIPLIERS

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Some conditions for distributivity of multiplication with respect to set addition are determined. In particular, distributivity holds if positive numbers are used, but does not hold for negative numbers in radix complement form. Use of simple recoding methods to achieve distributivity is not helpful, since costs of elementary multipliers and summing networks are increased. Distributivity does hold for the digit sets used for signed-digit arithmetic, and the requirement for distributivity provides guidance for the design of elementary multipliers.

## 1. Introduction

The first purpose of this paper is to point out that multiplication is not necessarily distributive with respect to set addition, and to determine some of the conditions for which distributivity does hold. The basic question is whether or not  $C \times (A + B) = (C \times A) + (C \times B)$ .  $A$ ,  $B$ , and  $C$  are digit sets, or sequences of consecutive integers including zero. The parentheses indicate the order of the operations.

The three conditions of practical importance for which distributivity holds are these:

1. All digits of both  $A$  and  $B$ , or all digits of  $C$  are positive or zero.
2. All digits of both  $A$  and  $B$ , or all digits of  $C$  are negative or zero.
3. The digit sets of any two of  $A$ ,  $B$ , and  $C$  are symmetric; that is, if  $X$  is an element of a digit set, so is  $-X$ .

Although other conditions for which distributivity holds are known to exist, it should not be assumed that distributivity holds in general when digit sets with both positive and negative values are involved.

The second purpose of this paper is to relate these results to the design of array multipliers. For the purposes of this paper, the simplistic assumption is made that the array multiplier consists of two more or less independent parts; the array of elementary multipliers, and the summing network which converts the array output to the form desired for the product.

For the binary multiplier array consisting of AND gates, distributivity holds if both multiplier and multiplicand are positive, but does not hold if negative values are introduced through use of a radix (two's) complement representation of operands. This provides a partial explanation of the fact that no reduction in the cost of the summing network can easily be achieved, even though the product precision is reduced by one bit. The simple attempts to achieve distributivity through recoding are self-defeating, since they require more complex elementary multipliers, and require a more costly summing network as well.

For signed digit arithmetic, which requires the use of symmetric digit sets, distributivity is easily achieved. The fact that distributivity must hold for elementary multipliers provides guidance as to the nature of logical designs which are permissible.

## 2. Terminology and Notation

The simplest example for which distributivity does not hold is presented to introduce the terminology and notation. The arithmetic operations involve digit sets, for example,  $a^0 = \{0, 1\}$ ,

$$a^1 = \{\bar{1}, 0\}, \quad b^1 = \{\bar{1}, 0, 1\}, \quad \text{and} \quad d^2 = \{\bar{2}, \bar{1}, 0, 1, 2\}.$$

The set addition  $b^1 + a^1 + a^0$  is illustrated by

	$a^0$
	$\begin{array}{ c c } \hline 0 & 1 \\ \hline \end{array}$
$a^1$	$\begin{array}{ c c } \hline \bar{1} & 0 \\ \hline \end{array}$
	$\begin{array}{ c c } \hline 0 & 1 \\ \hline \end{array}$

	$b^1$
	$\begin{array}{ c c c } \hline \bar{1} & 0 & 1 \\ \hline \end{array}$
$a^1$	$\begin{array}{ c c c } \hline \bar{1} & 0 & \bar{1} \\ \hline \end{array}$
	$\begin{array}{ c c c } \hline 0 & 0 & 0 \\ \hline \end{array}$

	$b^1$
	$\begin{array}{ c c c } \hline \bar{1} & 0 & 1 \\ \hline \end{array}$
$a^0$	$\begin{array}{ c c c } \hline 0 & 0 & 0 \\ \hline \end{array}$
	$\begin{array}{ c c c } \hline 1 & \bar{1} & 0 & 1 \\ \hline \end{array}$

Figure 1.

$$b^1 + a^1 + a^0$$

Figure 2.

$$b^1 + a^1 \times b^1$$

Figure 3.

$$b^1 + a^0 \times b^1$$

the addition table of Figure 1. The multiplicative

operations  $b^1 + a^1 \times b^1$  and  $b^1 + a^0 \times b^1$  are illustrated by the multiplication tables of Figures 2 and 3. For the example, the addition table

$d^2 + b^1 + b^1$  (Figure 4) and the multiplication

table  $b^1 + b^1 \times b^1$  (Figure 5) are also needed.

	$b^1$					$b^1$			
	$\bar{1}$	0	1			$\bar{1}$	0	1	
	$\bar{1}$	$\bar{2}$	$\bar{1}$	0		$\bar{1}$	1	0	$\bar{1}$
$b^1$	0	$\bar{1}$	0	1	$b^1$	0	0	0	0
	1	0	1	2		1	$\bar{1}$	0	1

Figure 4  
 $d^2 + b^1 + b^1$

	$b^1$					$b^1$			
	$\bar{1}$	0	1			$\bar{1}$	0	1	
	$\bar{1}$	1	0	$\bar{1}$		$\bar{1}$	1	0	$\bar{1}$
$b^1$	0	0	0	0	$b^1$	0	0	0	0
	1	$\bar{1}$	0	1		1	$\bar{1}$	0	1

Figure 5  
 $b^1 + b^1 \times b^1$

Consider the operations  $b^1 \times (a^1 + a^0)$ . If the addition is performed first, the results are  $b^1 + a^1 + a^0$  (Figure 1) followed by  $b^1 + b^1 \times b^1$  (Figure 5). If the multiplication is given precedence, the results are  $b^1 + b^1 + (b^1 \times a^1) + (b^1 \times a^0)$  (Figures 2 and 3) followed by  $d^2 + b^1 + b^1$  (Figure 4). Distributivity clearly does not hold.

A digit set is more generally designated by  $\triangle_{\delta}^{\omega}$ , and is characterized by two integer

parameters, the diminished cardinality  $\delta$  and the offset  $\omega$ . The digit set is a sequence of consecutive integers, including 0, ranging in value from  $-\omega$  to  $+(\delta-\omega)$ . A digit set is called normalized if  $\omega = 0$ , is called symmetric if  $\delta = 2\omega$ , (hence  $\delta$  is even), and is called negative normalized if  $\delta = \omega$ . For example, if  $\delta = 2$ , the digit set  $\{0, 1, 2\}$ , with  $\omega = 0$ , is

normalized; the digit set  $\{\bar{1}, 0, 1\}$ , with  $\omega = 1$ , is symmetric; and the digit set  $\{\bar{2}, \bar{1}, 0\}$  with  $\omega = 2$ , is negative normalized.

This terminology was chosen because  $\delta$  and  $\omega$  are preserved under set addition, for example, if

$$\triangle_{\delta_3}^{\omega_3} + \triangle_{\delta_1}^{\omega_1} + \triangle_{\delta_2}^{\omega_2}$$

then  $\delta_3 = \delta_1 + \delta_2$  and  $\omega_3 = \omega_1 + \omega_2$ .

It is convenient, for digit sets of lower cardinality, to adopt a less cumbersome notation by replacing  $\triangle_1$  by a,  $\triangle_2$  by b,  $\triangle_3$  by c, etc. The simplest symmetric digit set  $b^1 = \{\bar{1}, 0, 1\}$  will be of importance in sections to follow.

It may be noted that many of the structures for digital computer addition and subtraction, such as binary full adders and subtractors, conform to the rules of set addition; the symbol  $+$  is used to differentiate between inputs and outputs of such structures.

### 3. Basic Derivations

For the investigation of the existence of distributivity, define

$$\triangle_{\delta_4}^{\omega_4} + \triangle_{\delta_3}^{\omega_3} \times \left[ \triangle_{\delta_1}^{\omega_1} + \triangle_{\delta_2}^{\omega_2} \right] \quad 3.1$$

$$\triangle_{\delta_5}^{\omega_5} + \left[ \triangle_{\delta_3}^{\omega_3} \times \triangle_{\delta_1}^{\omega_1} \right] + \left[ \triangle_{\delta_3}^{\omega_3} \times \triangle_{\delta_2}^{\omega_2} \right] \quad 3.2$$

in which, for example, the product set

$$\triangle_{\delta_3}^{\omega_3} \times \triangle_{\delta_1}^{\omega_1} \text{ is found by}$$

$$\{-\omega_3, -\omega_3 + 1, \dots, \bar{1}, 0, 1, \dots, \delta_3 - \omega_3\} \times$$

$$\{-\omega_1, \dots, \bar{1}, 0, 1, \dots, \delta_1 - \omega_1\}$$

$$= \{-\max [\omega_1 (\delta_3 - \omega_3), \omega_3 (\delta_1 - \omega_1)], \dots, \bar{1}, 0, 1, \dots,$$

$$\max [\omega_1 \omega_3, (\delta_1 - \omega_1) (\delta_3 - \omega_3)]\}$$

Note that, in general, a product set is not necessarily a digit set, since not all consecutive values may be included. For example, the product set  $\{0, 1, 2\} \times \{0, 1, 2\} = \{0, 1, 2, 4\}$ .

The definition of distributivity employed here is that the most negative and the most positive

$$\text{values of the sets } \triangle_{\delta_4}^{\omega_4} \text{ and } \triangle_{\delta_5}^{\omega_5} \text{ shall be identical, or that } \omega_4 = \omega_5 \text{ and}$$

$$\delta_4 - \omega_4 = \delta_5 - \omega_5. \quad 3.3$$

It is easy to determine that

$$\omega_4 = \max [\omega_3 (\delta_1 - \omega_1 + \delta_2 - \omega_2), (\delta_3 - \omega_3) (\omega_1 + \omega_2)] \quad 3.4$$

$$\omega_5 = \max [\omega_3 (\delta_1 - \omega_1), \omega_1 (\delta_3 - \omega_3)] + \max [\omega_3 (\delta_2 - \omega_2), \omega_2 (\delta_3 - \omega_3)] \quad 3.5$$

$$\delta_4 - \omega_4 = \max [\omega_3 (\omega_1 + \omega_2), (\delta_3 - \omega_3) (\delta_1 - \omega_1 + \delta_2 - \omega_2)] \quad 3.6$$

$$\delta_5 - \omega_5 = \max [\omega_3 \omega_1, (\delta_3 - \omega_3) (\delta_1 - \omega_1)] + \max [\omega_3 \omega_2, (\delta_3 - \omega_3) (\delta_2 - \omega_2)] \quad 3.7$$

#### 4. Some Specific Conditions for Distributivity

##### 4.1. Introduction

The three specific conditions which appear to be of greatest practicality will be discussed first. These are that distributivity exists if any two of the three digit sets

$\triangle_{\delta_1}^{\omega_1}$ ,  $\triangle_{\delta_2}^{\omega_2}$ , or  $\triangle_{\delta_3}^{\omega_3}$  are symmetric, or

if  $\triangle_{\delta_1}^{\omega_1}$  and  $\triangle_{\delta_2}^{\omega_2}$  are both normalized or both negative normalized, or if  $\triangle_{\delta_3}^{\omega_3}$  is either

normalized or negative normalized. It is sufficient to show that symmetry applies simultaneously to  $\triangle_{\delta_1}^{\omega_1}$  and  $\triangle_{\delta_3}^{\omega_3}$ , since commutativity of set addition then implies that the result will hold for  $\triangle_{\delta_2}^{\omega_2}$  and  $\triangle_{\delta_3}^{\omega_3}$ .

##### 4.2 Other Conditions for Distributivity

A number of other conditions for distributivity have been found, primarily by a systematic search of examples, after elimination of cases previously found. These include:

a)  $\triangle_{\delta_1}^{\omega_1} = \triangle_{\delta_2}^{\omega_2}$

b)  $\triangle_{\delta_1}^{\omega_1} = -\triangle_{\delta_3}^{\omega_3}$  if either  $\omega_1 = 0$  or  $\omega_3 = 0$

c)  $\triangle_{\delta_1}^{\omega_1} = -\triangle_{\delta_3}^{\omega_3}$

if  $\delta_1 - \omega_1 > \omega_1$  and  $(\delta_1 - \omega_1)(\delta_2 - \omega_2) > \omega_1 \omega_1$

or  $\omega_1 > \delta_1 - \omega_1$  and  $\omega_1 \omega_2 > (\delta_1 - \omega_1)(\delta_2 - \omega_2)$

d)  $\delta_3 = 2\omega_3$  if  $\delta_1 - \omega_1 > \omega_1$  and  $\delta_2 - \omega_2 > \omega_2$   
or  $\omega_1 > \delta_1 - \omega_1$  and  $\omega_2 > \delta_2 - \omega_2$

e)  $\delta_1 + \delta_2 = 2(\omega_1 + \omega_2)$  and  $\omega_3 = 0$  or  $\delta_3 = \omega_3$

It is anticipated that a continuation of the systematic search will reveal more examples, with increasingly complex conditions for their applicability. Whether the results will be merely an intellectual curiosity or of practical value is yet to be determined.

#### 5. Examples of Simple Array Multipliers

A set of simple examples of array multipliers indicates the nature of the problem in a practical situation. Consider first the multiplication of the 3 bit positive multiplicand  $4x_2 + 2x_1 + x_0$  by

the 3 bit positive multiplier  $4y_2 + 2y_1 + y_0$ , with

the multiplier and multiplicand each ranging from 0 to +7. The product set will range, as expected, from 0 to 49, and may be formed by summing the properly weighted outputs of a 3x3 array of elementary multipliers, which are AND gates, as indicated in Figure 5.1.

	$4x_2$	$2x_1$	$x_0$
$4y_2$	16	8	4
$2y_1$	8	4	2
$y_0$	4	2	1

Figure 5.1 Weights for a 3x3 positive array of AND gates

It is easy to verify that the maximum sum is  $16 + 2(8) + 3(4) + 2(2) + 1 = 49$  as expected.

Alternatively, consider the 3x3 array multiplier when a twos (or radix) complement representation is employed, i.e., the multiplicand is  $-4x_2 + 2x_1 + x_0$  and the multiplier is  $-4y_2 + 2y_1 + y_0$ .

Each operand ranges from -4 to +3, hence the product set should range from -12 to +16. Consider Figure 5.2, indicating the weights for the 3x3 AND gate array.

	$-4x_2$	$2x_1$	$x_0$
$-4y_2$	+16	-8	-4
$2y_1$	-8	+4	+2
$y_0$	-4	+2	+1

Figure 5.2 Weights for a 3x3 radix complement array of AND gates

The summing network must be able to accommodate values ranging from the most negative value  $-8 + (-8) + (-4) + (-4) = -24$  to the most positive value  $16 + 4 + 2 + 2 + 1 = 25$ . Obviously, distributivity does not apply.

#### 6. Implications on the Design of Array Multipliers for Radix Complement Representations

A radix complement representation can be transformed into a symmetric representation by a generalization of the Booth (or differentiating) recoding.

$$\text{Let } y = -2^5 y_5 + 2^4 y_4 + 2^3 y_3 + 2^2 y_2 + 2 y_1 + y_0.$$

$$\text{Then } 2y = -2^6 y_5 + 2^5 y_4 + 2^4 y_3 + 2^3 y_2 + 2^2 y_1 + 2 y_0 + y_{-1}, \text{ with } y_{-1} = 0.$$

$$\text{Also } -2^5 y_5 = -2^6 y_5 + 2^5 y_5.$$

$$\begin{aligned} y = 2y - y &= 2^6 (-y_5 + y_5) + 2^5 (y_4 - y_5) \\ &+ 2^4 (y_3 - y_4) + 2^3 (y_2 - y_3) + 2^2 (y_1 - y_2) \\ &+ 2 (y_0 - y_1) + (y_{-1} - y_0) \end{aligned}$$

In the binary form shown, the resulting recoding is the Booth or differentiating recoding, and transforms each binary digital position into a redundant symmetric digit set  $b^1 = \{\bar{1}, 0, 1\}$ .

The above formulation may also be written as

$$\begin{aligned} &2^4 (-2y_5 + y_4 + y_3) + 2^2 (-2y_3 + y_2 + y_1) \\ &+ 2^0 (-2y_1 + y_0 + y_{-1}) \end{aligned}$$

Each radix 4 digit is then a member of the symmetric digit set  $d^2 = \{\bar{2}, \bar{1}, 0, 1, 2\}$ .

From a theoretical point of view, the use of symmetric digit sets is apparently advantageous. First, distributivity is preserved, as indicated in section 4. Second, for a normalized digit set of diminished cardinality  $\delta$  ( $\delta$  even), the diminished cardinality of the product set is  $\delta^2$ . In contrast, for a symmetric digit set with  $\delta = 2\omega$ , the product set ranges from  $-\delta^2/4$  to  $\delta^2/4$ , with diminished cardinality  $\delta^2/2$ .

For the purposes of this paper, an  $n \times n$  array multiplier will be considered to consist of two parts; an array of elementary multipliers, and a summing network which converts the weighted outputs of the multiplier array to the desired weighted binary product.

Current practice is to use an array of  $n^2$  AND gates for the elementary multipliers, with each AND gate performing an elementary multiplication of

one of the types  $a^0 + a^0 \times a^0 = a^1 \times a^1$ , or

$a^1 + a^1 \times a^0 = a^0 \times a^1$ . The diminished cardinality of the weighted sum of the AND gate array will

always be  $2^{2n} - 2^{n+1} + 1$ .

Although the use of symmetric digit sets appears to be theoretically attractive, the recoding procedures necessary to achieve them increase the information content. For example, for the Booth recoding, each digital position requires a

type  $b^1$  digit set, and each element of the multiplier array performs the operation  $b^1 + b^1 \times b^1$ , at

a cost roughly equivalent to a half adder. The

information from each of the  $n^2$  elementary multipliers is also doubled, which increases the cost of the summing network as well. Thus, use of the Booth recoding is clearly not cost effective.

Use of the radix four symmetric digit set

$2a^1 + b^0$  requires the use of  $n^2/4$  elementary multipliers of the form  $4a^1 + 2a^0 + b^0 + (2a^1 + b^0) \times$

$(2a^1 + b^0)$ . The output requires 4 bits, so that the information presented to the summing network is  $4(n^2/4) = n^2$ , which is the same as that of the

conventional array. Thus the summing network should require roughly the same amount of hardware as the conventional array, but each elementary multiplier is more complicated than the four AND gates of the conventional array that it replaces.

It is also possible to compare the array version of the serial multiplier used in Illiac II and Illiac III. The multiplicand

$$x = -2^{n-1} x_{n-1} + \sum_{j=0}^{n-2} 2^j x_j \text{ was left in}$$

conventional form and the multiplier  $y$  was recoded radix 4, with each digit of the form

$$d^2 = \{\bar{2}, \bar{1}, 0, 1, 2\}. \text{ Thus } y = 2y - y =$$

$$\sum_{i=0}^{1/2(n-2)} (-2y_{2i+1} + y_{2i} + y_{2i-1}), \text{ with } y_{-1} = 0.$$

The recoded digit of the form  $d^2$  controls a conditional complementing and doubling circuit.

Without going into the design details, the order of magnitude costs are about the same as

those of the conventional array of  $n^2$  AND gates.

The reduced cost of the summing network approximately compensates for the additional costs of the multipliers and the conditional complementing and doubling circuits.

## 7. Conclusions

For conventional representations of numbers, the results obtained are disappointing. For the standard  $n \times n$  array of AND gates, the diminished cardinality of the array output remains invariant when the weights of some of the digits are made negative, and no advantage can be taken of the reduced cardinality which theoretically could be achieved.

Attempts to convert from conventional to symmetric digit sets are self-defeating, because of the increase in information content and because of the increased complexity of both the multiplier array and the summing network.

The technique of recoding the multiplier radix 4, with conditional complementation and doubling, appears to deserve further study in detail, since the order of magnitude calculations indicate that it is cost competitive with the conventional  $n \times n$  AND gate array.

For signed-digit arithmetic, the results obtained reinforce the desirability of using symmetric digit sets. Not only are symmetric digit sets advantageous for subtraction, since  $x$  and  $-x$  are always members of the set, but the results indicate that symmetric digit sets are most efficient for multiplication.

For symmetric array multipliers, the tradeoff between the cost of the elementary multiplier array and the cost of the summing network deserves investigation. It appears to be advantageous to

use  $n^2/2$  elementary multipliers of the form

$$2a^1 + b^0 + b^1 \times (2a^1 + b^0) \text{ rather}$$

than  $n^2$  multipliers of the form  $b^1 + b^1 \times b^1$ ,

because of the reduced costs of the summing

network. Use of  $n^2/4$  elementary multipliers of the

$$\text{form } 4a^1 + 2a^0 + b^0 + (2a^1 + b^0)$$

$\times (2a^1 + b^0)$  should also be investigated.

Note that the designs of elementary multipliers must be logical designs. For example  $b^1 \times$

$$(2a^1 + b^0) \text{ should not be designed as } (b^1 \times 2a^1) +$$

$(b^1 \times b^0)$ , since distributivity does not hold. The results obtained thus indicate which algebraic manipulations are permissible for efficient multiplier design.

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