FLOATING-POINT RECURRING RATIONAL ARITHMETIC SYSTEM

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ABSTRACT

Major computer arithmetic systems are based on the concept of realizing only terminate rationals in positional notation. This paper proposes a new arithmetic scheme of indicating periodicity in the radix representation of a mantissa to realize recurring rationals as well as terminate rationals. A new arithmetic system adopting the scheme, called the "FLP/R* arithmetic system", is proposed. Properties of the FLP/R* numbers and the procedure of the FLP/R* arithmetic are described.

1. Introduction

In most of the existing computer arithmetic systems, numbers are represented in the positional notation of the fixed-point form (FXP) or the floating-point form (FLP). Owing to their property that the range of representable numbers is wide, the FLP arithmetic systems are widely used. Given a base b, any real number x is expressed as follows:

$$x = M * b^E$$

This is called the "FLP form", where M is called the "mantissa" and E the "exponent". In the conventional FLP arithmetic systems of finite precision, only terminate rationals have been realized. The computational accuracy of these systems concerning the number of significant and guard digits and the rounding method for the guard digits has been argued [15-17].

For the purpose of performing error-free computation, rational arithmetic using fractional notation [3-7] or p-adic (Hensel code) notation [10-14] has been studied. In finite precision, a rational arithmetic needs some expensive processes for approximation.

This paper proposes a new arithmetic scheme to unify conventional FLP arithmetic and rational arithmetic. In this scheme, the periodicity of the radix representation of a mantissa is indicated in order to realize recurring as well as terminate rationals.

The characteristics of each notation are discussed in Chapter 2. In Chapter 3, the new arithemetic scheme of indicating periodicity of the radix representation of a mantissa is proposed, which leads to a new arithmetic system, called the "FLP/R* arithmetic system". Static properties of the FLP/R* numbers are investigated in Chapter 4. Procedures of the FLP/R* arithmetic are described in Chapter 5.

2. Notation and Arithmetic

2.1. Positional Notation

It is generally known that a real number is expressed as an infinite series of rational numbers. In the positional notation, a number x is represented using a sequence of coefficients, namely "digits" $\{d_i\}$ of an infinite power series expansion with a base (or radix) b as follows:

$$x = \ldots + d_2 * b^2 + d_1 * b^1 + d_0 + d_{-1} * b^{-1} + d_{-2} * b^{-2} + \ldots$$

$$= \cdots d_2 d_1 d_0 . d_{-1} d_{-2} \cdots {}_{(b)}$$

where $\cdots d_2 d_1 d_0 . d_{-1} d_{-2} \cdots {}_{(b)}$ is called the "radix representation". Imaginary numbers, irrational numbers, or negative integers could be used for the base [2].

Here, let the base be a positive integer $b > 1$ as usual, then there exist two kinds of positional notation for real numbers [1,2];

1) right-expanding notation

$$x = M * b^E \ , \ |M| = b^{-1} * (d_{-1} + b^{-1} * (d_{-2} + \ldots)) \quad , \ d_{-1} \neq 0$$

2) left-expanding notation

$$x = I * b^F \ , \ I = ((\ldots + d_2) * b + d_1) * b + d_0 \quad , \ d_0 \neq 0$$

2.1.1. Property of Expansion

The expansion in positional notation for a number possesses one of the following three properties: termination, periodicity, and irregularity.

(1) Termination

Rational numbers whose denominator has only factors to divide the base have terminate

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expansions.

$$|M| = .d_{-1}d_{-2}\cdots d_{-n} \underset{(b)}{=} \frac{\sum_{i=-1}^{-n} d_i * b^{i+n}}{b^n}$$

$$I = d_{h-1}d_{h-2}\cdots d_0 \underset{(b)}{=} \sum_{i=0}^{h-1} d_i * b^i$$

(2) Periodicity

The expansion of a rational number whose denominator has any factor prime to the base exhibits periodicity.

$$|M| = .d_{-1}\cdots d_{-n}\overline{d_{-n-1}\cdots d_{-n-r}} \underset{(b)}{=} \frac{\sum_{i=-1}^{-n-r} d_i * b^{i+n+r} - \sum_{i=-1}^{-n} d_i * b^{i+n}}{b^n * (b^r - 1)}$$

$$I = \underline{d_{h+r-1}\cdots d_h}d_{h-1}\cdots d_0 \underset{(b)}{=} \frac{\sum_{i=0}^{h-1} d_i * b^{i+r} - \sum_{i=0}^{h+r-1} d_i * b^i}{b^r - 1}$$

where the overline or the underline denotes a recurring clause.

$$|M| = \sum_{i=-1}^{-n-r} d_i * w_i \ , \ w_i = \begin{cases} b^i & ; -1 \geq i \geq -n \\ \dfrac{b^{i+r}}{b^r - 1} & ; -(n+1) \geq i \geq -(n+r) \end{cases}$$

$$I = \sum_{i=0}^{h+r-1} d_i * w_i' \ , \ w_i' = \begin{cases} -\dfrac{b^i}{b^r - 1} & ; h+r-1 \geq i \geq h \\ b^i & ; h-1 \geq i \geq 0 \end{cases}$$

Note that if each digit in the recurring clause is equal to (b-1), then the expansion terminates essentially.

$$\sum_{i=-n-1}^{-\infty} (b-1) * b^i = b^n \quad ; \text{right-expansion}$$

$$\sum_{i=h}^{\infty} (b-1) * b^i = -b^h \quad ; \text{left-expansion}$$

Thus, both notations are equivalent for rational numbers. A representation of a rational number in one notation can be converted into that of the other notation. For example,

$$x = .1234\overline{567} * 10^0 \quad \text{(right-expansion)}$$
$$= \underline{243}3667 * 10^{-4} \quad \text{(left-expansion)}$$

(3) Irregularity

The irrational numbers have irregular expansions.

2.1.2. Approximation

In a positional system of finite precision, only terminate rationals are representable and the arithmetic requires an approximation process. In the right-expanding notation, the approximation process consists of only a truncation and rounding. For example,

$$x = .1234\overline{567} * 10^0 \dashrightarrow .1234568 * 10^0$$

The computation accuracy has been improved by increasing the number of significant digits, attaching guard digits, and providing a variety of rounding methods [16-19].

In the left-expanding notation, however, the operation result of a simple truncation could turn into an absurd value. For example,

$$x = \underline{243}3667 * 10^{-4} \text{ -X-> } 2433667 * 10^{-4}$$

E.V.Krishnamurthy has proposed a "finite-segment p-adic arithmetic" for the left-expansion of finite precision [10,11]. The arithmetic is based on the residue arithmetic under the assumption that each weight is fixed, and needs the approximation process of solving a complex diophantine equation. As a more practical method of this process, T.M.Rao and R.T.Gregory proposed an algorithm based on the table look-up procedure [12], but the higher the precision, the larger the table needed will be.

One of the most characteristic features of p-adic arithmetic is its exact computation when the basic arithmetic operations proceeds uniformly from the LSD to the MSD. E.C.R. Hehner and R.N.S.Horspool have proposed an arithmetic system indicating the periodicity on the left-expansion of infinite precision [13,14]. We have investigated how the non-recurring clause and recurring clause grow by each exact arithmetic operation for both positional notation, and found that the growth could be too rapid to be supported persistently [1].

2.2. Fractional Notation

A rational is defined as a number that is expressed with a pair of integers as follows.

$$x = u/v \quad ; \quad u, v \in Z \ , \ v \neq 0$$

where Z denotes the set of the integers.
Such notation is called the "fractional notation".

2.2.1. Reduction

In fractional arithmetic, the numerator and denominator may grow to be quite large. Reduction is the process which generates a simple fraction whose numerator and denominator are relatively prime in order to minimize the growth. The reduction procedure is usually carried out with a binary GCD algorithm [6,7].

2.2.2. Approximation

If precision is limited, fractional arithmetic also needs an approximation process.

D.W.Matula and P.Kornerup have proposed "fixed-slash and floating-slash arithmetic" for finite precision, which needs the "mediant rounding" approximation method [2-6]. The rounding procedure truncates the continued fraction expansion of the original fraction at some step appropriate to the precision limitation with a Euclidian algorithm.

Both the binary GCD algorithm and Euclidian algorithm contain sequential processes of $\mathcal{O}(n)$ where n is the precision [2].

3. Proposal of FLP/R* Arithmetic System

3.1. Periodicity Indication

An approach in which the conventional FLP arithmetic on the right-expansion and the rational arithmetic for highly accurate computation is mixed is found in [8]: the combination made in unifying these arithmetics realizes recurring rationals at a low cost.

A new arithmetic scheme proposed here is to indicate the periodicity of the right-expansion in order to realize recurring as well as terminate rationals. Concretely, we propose a new arithmetic system adopting this scheme, called the "FLP/R* arithmetic system". (FLP/R* is an abbreviation of FLoating-Point Recurring Rationals.)

3.2. Recurring Rational

If the denominator of a number contains any factor that is relatively prime to the base, then the radix representation of the number possesses periodicity. Therefore, a number may belong to recurring rationals in a certain base, and to terminate rationals in another base. The set of the recurring rationals is determined to the base.

Let us assume that a number x is a recurring rational within $0<x<1$ in a base b. Then the number x is expressed as follows:

$$x = u/v = \frac{\sum_{i=-1}^{-n-r} d_i * b^{i+n+r} - \sum_{i=-1}^{-n} d_i * b^{i+n}}{b^n * (b^r - 1)} = .d_{-1} \cdots d_{-n} \overline{d_{-n-1} \cdots d_{-n-r}}_{(b)}$$

It is found that there exists the following relation among the denominator v , the length n of the non-recurring clause, and the period r of the recurring clause.

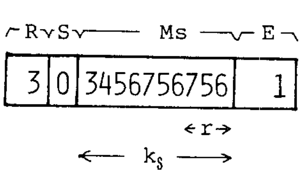
$$v = v_1 * v_2$$
$$\gcd(v_1, v_2) = 1$$
$$v_1 \mid b^n, \quad v_2 \mid (b^r - 1) \qquad (*)$$

The period r is the smallest integer to satisfy that

$$(b^r - 1) \bmod v_2 = 0$$

3.3. Representation of FLP/R* Number

The FLP/R* arithmetic system is realized by attaching an additional indicator, the "R-indicator", to an FLP arithmetic system where the length of the mantissa field is fixed, as shown in Fig.1. The R-indicator indicates the period of the recurring clause in the radix representation of the mantissa. The conventional mantissa part, that is, the radix representation of a mantissa, is called the "S-mantissa". ("S-" means "Significant".) Thus a mantissa is represented by the R-indicator and the S-mantissa.



S : Sign
Ms: Significant Mantissa
R : Recurring Indicator
E : Exponent

Fig.1 Representation of a FLP/R* number
(The example shows $.34567*10^1$)

* d | m implies that d is a divisor of m.

Let the base be b and the S-mantissa consist of k_s digits. Then there could exist rational numbers whose recurring clause is r=0 to k_s digits long. The maximum period to be provided, that is, the maximum value of the R-indicator, can be set in the range $0 \le r_{max} (\equiv k_g) \le k_s$.

Thus, the FLP/R* mantissa requires t bits for the R-indicator in addition to t bits for the S-mantissa as follows:

$$t = t_R + t_M$$
$$t_R = \lceil \log_2(k_g + 1) \rceil$$
$$t_M = \begin{cases} k_s * \lceil \log_2 b \rceil & ; \text{unpacked representation} \\ \lceil k_s * \log_2 b \rceil & ; \text{packed representation} \end{cases}$$

4. Property of FLP/R* Numbers

4.1. Set of FLP/R* Mantissas

In ordinary FLP systems, the set of the FLP mantissas consists of only terminate rationals. Let $R_T^{k_s}$ denote the set of the terminate rationals on the k_s-digit S-mantissa, then

$$R_T^{k_s} = \{x \mid 0<x=u/v<1 , v \mid b^{k_s}\} .$$

In the proposed FLP/R* system, the set of the FLP/R* mantissas consists of terminate rationals and recurring rationals. Let $R_U^{(k_s,k_g)}$ denote the set of the FLP/R* mantissas with the k_s-digit S-mantissa and up to k_g-digit period. Then

$$R_U^{(k_s,k_g)} = R_T^{k_s} \cup R_R^{(k_s,k_g)}$$

where

$R_R^{(k_s,k_g)}$ denotes the set of recurring rationals with the k_s-digit S-mantissa and up to k_g-digit period, and is expressed as

$$R_R^{(k_s,k_g)} = R_{R_1}^{(k_s,k_g)} \cup R_{R_2}^{(k_s,k_g)} \cup \dots \cup R_{R_{k_g}}^{(k_s,k_g)} = \bigcup_{j=1}^{k_g} R_{R_j}^{(k_s,k_g)}$$

$R_{R_j}^{(k_s,k_g)}$ denotes the set of recurring rationals whose period is j, and is expressed as

$$R_{R_j}^{(k_s,k_g)} = \{x \mid 0<x=u/v<1 , v \mid \nu_j = b^{k_s-j} * (b^j - 1)\}$$

The above relations are rewritten using sine functions. Let a function $f_j(x)$ be

$$f_j(x) = \sin(\pi * \nu_j * x) \; ; \; j=0,1,2,\dots,k_g$$

where

$$\nu_j = \begin{cases} b^{k_s} & ; j=0 \\ b^{k_s-j} * (b^j - 1) & ; j=1,2,\dots,k_g \end{cases}$$

Furthermore, let a function f(x) be

$$f(x) = \prod_{j=0}^{k_g} f_j(x)$$

Then the sets $R_T^{k_s}$, $R_R^{(k_s,k_g)}$, and $R_U^{(k_s,k_g)}$ are expressed as

$$R_T^{k_s} = \{x \mid f_0(x)=0 \; ; \; 0<x<1\} ,$$
$$R_R^{(k_s,k_g)} = \{x \mid \prod_{j=0}^{k_g} f_j(x)=0 \; ; \; 0<x<1\} , \text{ and}$$
$$R_U^{(k_s,k_g)} = \{x \mid f(x)=0 \; ; \; 0<x<1\} .$$

An example of the function f(x) for b=2 and $k_s=k_g$ obtained is shown in Fig.2, and examples of the distribution of the FLP/R* mantissas are shown in Fig.3.

4.2. Number of FLP/R* Mantissas

Each element of the set $R_{||}^{(k_s,k_g)}$ is a rational if the denominator is a divisor of anyone of ν_j; $j=0,1,2,\ldots,k_g$. Then the number of the elements is expressed as follows:

$$n(R_{||}^{(k_s,k_g)}) = \sum_{d\in D} \phi(d)$$

where

$\phi(d)$: "Euler's function", defined for any positive integer d as the number of positive integers not exceeding d that are relatively prime to d,

$$D = \bigcup_{j=0}^{k_g}\{d \mid d \text{ is a divisor of } \nu_j\}.$$

Examples of the number $n(R_{||}^{(k_s,k_g)})$ for b=2,16 and $k_s=k_g$ obtained are shown in Fig.4. In this figure, the difference $d_1=n(R_{||}^{(k_s,k_g)})-n(R^{k_s})$ is the number of recurring rationals added to the ordinary FLP mantissas in the FLP/R* system, and the difference $d_2=n(R^{k'})-n(R_{||}^{(k_s,k_g)})$, where $k'=k_s+\lceil\log_2(k_g+1)\rceil$, includes the redundancy of the FLP/R* representation.

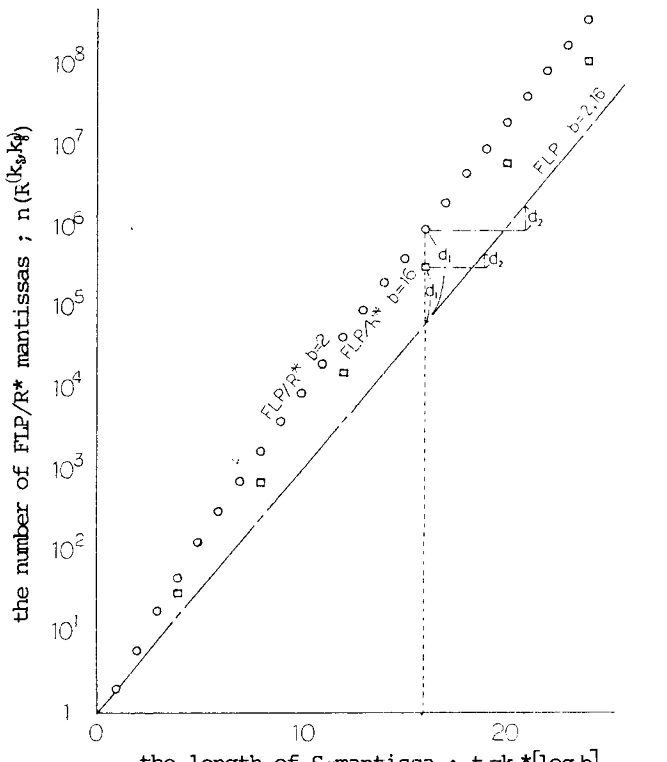


Fig.4 The Number of FLP/R* Mantissas ($k_s=k_g$)

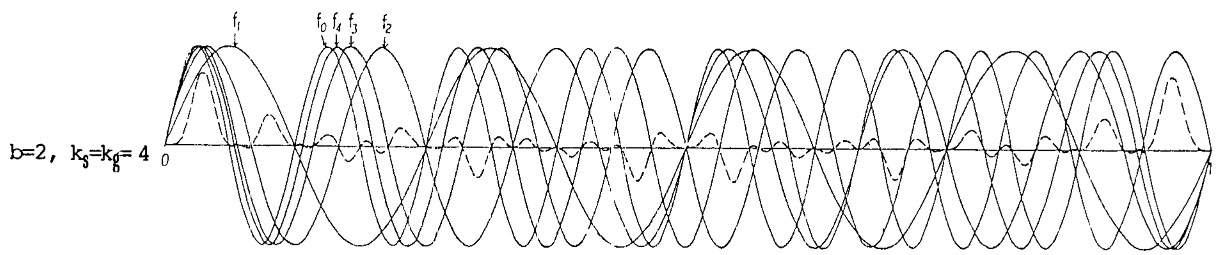


Fig.2 Distribution Function $f(x)=\prod_{j=0}^{k_g} f_j(x)$

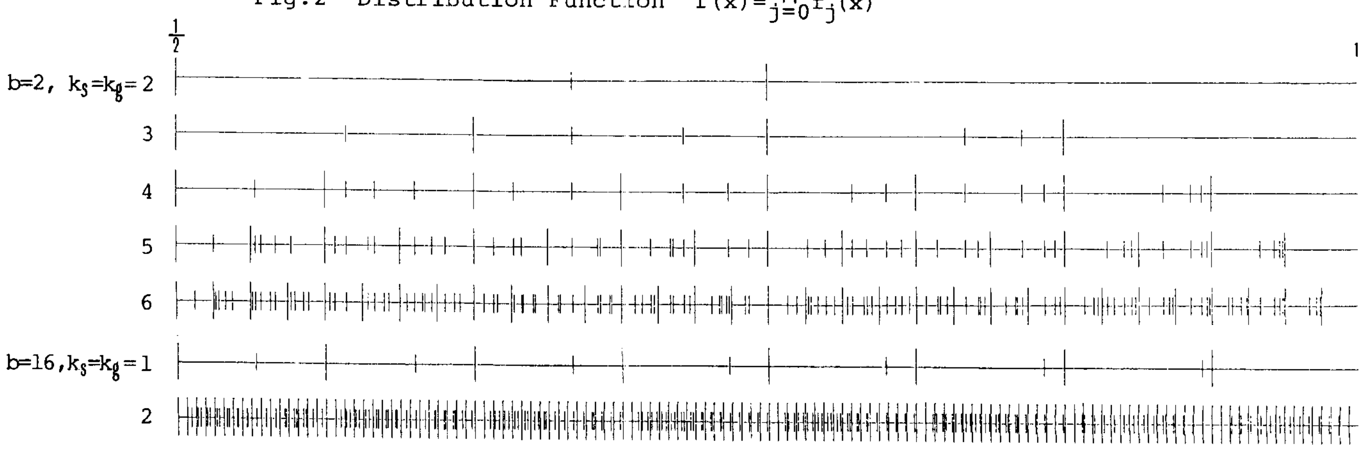


Fig.3 Distribution of FLP/R* Mantissas

4.3. Gap Distribution of FLP/R* Mantissas

As shown in Fig.2 and Fig.3, the distribution of FLP/R* mantissas is not uniform.

Let g be the gap between the adjacent FLP/R* mantissas M_1 and M_2 such that

$$M_1 = u_1/v_1 \ , \ M_2 = u_2/v_2 \ , \ M_1 < M_2$$

Then g is expressed as

$$g = M_2 - M_1 = \frac{u_2 * v_1 - u_1 * v_2}{v_1 * v_2}$$

As mentioned before, each of v_1 and v_2 is a divisor of anyone of ν_j ; $j=0,1,\ldots,k_g$. Therefore, there exist the g's minimum g_{min} and maximum g_{max} as follows:

$$g_{min} = \frac{1}{b^{k_s} * (b^{k_g}-1)}$$

where $[M_1,M_2]=[\frac{1}{b^{k_s}}, \frac{1}{b^{k_s-k_g} * (b^{k_g}-1)}], [\frac{b^{k_s-k_g} * (b^{k_g}-1)-1}{b^{k_s-k_g} * (b^{k_g}-1)}, \frac{b^{k_s}-1}{b^{k_s}}]$

$$g_{max} = \frac{1}{b^{k_s}}$$

where $[M_1,M_2]=[0,\frac{1}{b^{k_s}}], [\frac{b^{k_s}-1}{b^{k_s}}, 1]$

Furthermore, using the number of mantissas, $n(R_U^{(k_s,k_g)})$, the average of g's, g_{ave}, is obtained as

$$g_{ave} = \frac{1}{n(R_U^{(k_s,k_g)})}$$

An example of the gap distribution for b=2 and $k_s=k_g$ is shown in Fig.5.

5. Procedure of FLP/R* Arithmetic

The FLP/R* arithmetic system is based on the concept that indicating periodicity corresponds to adding guard digits to the S-mantissa.

The minimum gap between FLP/R* mantissas of $R_U^{(k_s,k_g)}$ is larger than the gap between FLP mantissas of $R_I^{k_s+k_g}$.

$$g_{min} = \frac{1}{b^{k_s} * (b^{k_g}-1)} > \frac{1}{b^{k_s+k_g}}$$

This shows that the FLP/R* mantissas can be distinguished from each other with (k_s+k_g)-digits precision. Then the FLP/R* arithmetic operation is performed so that the result can contain (k_s+k_g)-digits precision.

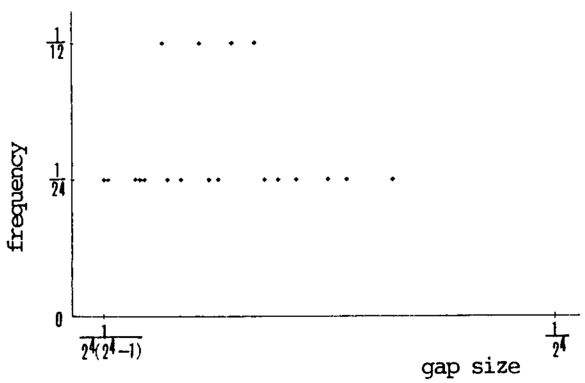


Fig.5 Gap Distribution $(b=2, k_s=k_g=4)$

The basic arithmetic operations in the FLP/R* system can be carried out with the conventional algorithm for ordinary FLP systems, and need the following two additional processes:

1) S-mantissa expansion as a preprocess, and
2) FLP/R* rounding as a postprocess.

We have implemented the simulator of an FLP/R* arithmetic system on a VAX-11/750 system. The simulator adopts the normalized sign/absolute form for the mantissa representation, and the digit-slice algorithm for the basic arithmetic operation.

5.1. S-mantissa Expansion

For an arithmetic operation, the operands in the memory are loaded into registers in the arithmetic unit. In order to obtain the operation result with (k_s+k_g)-digits precision, it is necessary that the S-mantissa of each operand is expanded using the information of its R-indicator. This process is called the "S-mantissa expansion", and consists of extracting the recurring clause, shifting, and adding it as illustrated in Fig.6. The lower part attached to the S-mantissa is called the "G-mantissa". ("G-" means "Guard".)

The length of the expansion in the normalized FLP/R* system is shown in Table 1. Note that the value for addition/subtraction is obtained with consideration with the maximum loss of digits $(=k_s+k_g-1)$ for adjacent operands with the minimum gap.

After this process, the arithmetic operation is performed and the operation result is normalized in (k_s+k_g)-digits precision.

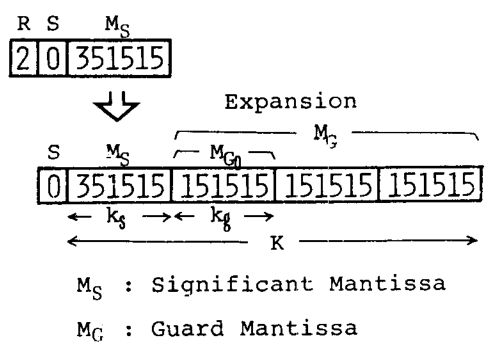


Fig.6 S-mantissa Expansion

Table 1. Expansion Length

	K_X	K_Y
X ± Y	$2*(k_s+k_g)$	$2*(k_s+k_g)$
X * Y	k_s+k_g	k_s+k_g
X / Y	$2*(k_s+k_g)$	k_s+k_g

5.2. FLP/R* Rounding

The operation result with (k_s+k_g)-digits precision needs to be approximated to the closest FLP/R* number, and then be represented with the R-indicator and the S-mantissa of k_s-digits precision. This process, called the "FLP/R* rounding", is illustrated in Fig.7.

Let a number be representable in the FLP/R* system, then the digit sequence of the G_0-mantissa depends on that of the S-mantissa. The rounding process consists of the following two procedures:

1) generating candidates $\{C_i\}$, representable numbers determined by the digit sequence of the S-mantissa of the operation result, and
2) choosing one out of the candidates $\{C_i\}$, whose G_0-mantissa is the closest to that of the result.

For the k_g-digit G_0-mantissa, there exist at most (k_g+2) candidates of two terminate rationals and k_g recurring rationals.

Let us assume that the operation result M is obtained as

$$M = .\underbrace{d_{-1}\cdots d_{-k_s}}_{M_S}\ \underbrace{d_{-k_s-1}\cdots d_{-k_s-k_g}}_{M_{G_0}}$$

Then the candidates $\{C_i\}$ are determined as follows:

Terminate rationals : C_{k_g+1}, C_0

$$C_{k_g+1} = 10\cdots0_{(b)} = b^{k_g}$$
$$C_0 = 0\cdots0_{(b)} = 0$$

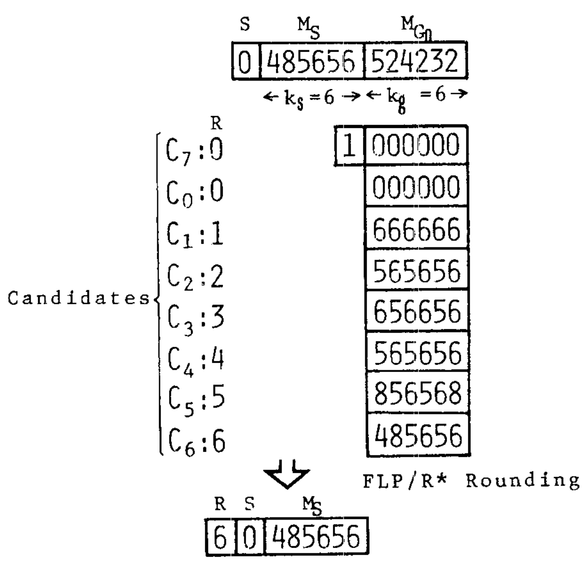


Fig.7 FLP/R* Rounding

Recurring rationals : $C_1, C_2, \ldots, C_{k_g}$

$$C_j = \overbrace{\underbrace{d_{-k_s+j-1}\, d_{-k_s+j-2} \cdots d_{-k_s}}_{j}\ d_{-k_s+j-1}\, d_{-k_s+j-2} \cdots}^{k_g}{}_{(b)}$$
$$= \sum_{i=-k_s-1}^{-k_s-k_g} d_{n_i} * b^{i+k_s+k_g}$$

where $n_i = -(k_s - j + (|k_s+1-i| \bmod j))$

The value of the G_0-mantissa is expressed as

$$M_{G_0} = d_{-k_s-1} \cdots d_{-k_s-k_g\,(b)} = \sum_{i=-k_s-1}^{-k_s-k_g} d_i * b^{i+k_s+k_g}$$

Therefore, for $0 \le j \le (k_g+1)$, the candidate C_j that has the smallest difference of $|M_{G_0}-C_j|$ is chosen, and the value $r=j \bmod (k_g+1)$ is stored in the R-indicator of the destination. If C_{k_g+1} is chosen, the LSD of the S-mantissa is incremented by 1.

The FLP/R* arithmetic possesses the property that the operation result tends to be equal to the true value through the cancellation of intermediate round-off errors. Examples of the FLP/R* arithmetic are shown in Fig.8.

```
***** FLP/R* ARITHMETIC *****

<initialization>
BASE=10 MsSIZE=5 MgSIZE=0

<execution>
+ 0 85 0 / + 0 163 0 = + 0 52147 0
+ 0 71 0 / + 0 489 0 = + 0 14519 0
+ 0 52147 0 + + 0 14519 0 = + 0 66666 0
+ 0 85 0 / + 0 163 0 = + 0 52147 0
+ 0 163 0 / + 0 255 0 = + 0 63921 0
+ 0 52147 0 * + 0 63921 0 = + 0 33333 0

<initialization>
BASE=10 MsSIZE=4 MgSIZE=4

<execution>
+ 0 85 0 / + 0 163 0 = + 4 5214 0
+ 0 71 0 / + 0 489 0 = + 0 1452 0
+ 4 5214 0 + + 0 1452 0 = + 1 6666 0
+ 0 85 0 / + 0 163 0 = + 4 5214 0
+ 0 163 0 / + 0 255 0 = + 1 6392 0
+ 4 5214 0 * + 1 6392 0 = + 1 3333 0

TRACE < expand end >
x : s=+ r=0 m=85 e=0 d=00085000000000000000000
y : s=+ r=0 m=163 e=0 d=00163000000000000000000
z : s=+ r=0 m=0 e=0 d=00000000000000000000000
 TRACE < div end >
x : s=+ r=0 m=85 e=-2 d=00000000010400000000000
z : s=+ r=0 m=0 e=-1 d=6-7-95-33-60-800000000000000
 TRACE < cnvusf end >
z : s=+ r=0 m=0 e=-1 d=52147239200000000000000
 TRACE < norm end >
z : s=+ r=0 m=0 e=0 d=05214723920000000000000
 TRACE < round end >
z : s=+ r=4 m=5214 e=0 d=05214723920000000000000

 + 0 85 0 / + 0 163 0 = + 4 5214 0
```

Fig.8 Examples of FLP/R* Arithmetic

6. Conclusion

In this paper, a new arithmetic scheme which indicates the periodicity in the right-expanding positional notation has been proposed, and the "FLP/R* arithmetic system" adopting the scheme has been described. The FLP/R* system is an union of a conventional positional system and a fractional system, in which indicating the periodicity has the effect of increasing the accuracy in numerical computations. The distribution of the FLP/R* mantissas is not uniform. The arithmetic in the proposed system needs 1) the preprocess of the "S-mantissa expansion" and 2) the postprocess of the "FLP/R* rounding" besides the basic operations in the ordinary FLP systems. As a result, the intermediate round-off errors in an operation tend to be canceled out. Hereafter, we will analyze the effective tendency of cancellation by the simulator implemented.

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