MIDDLE ERROR CORRECTION AND ADDITIVE OVERFLOW DETECTION
WITH MAGNITUDE INDICES IN RESIDUE CODE

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ABSTRACT

A new approach for correcting multiple errors and detecting an additive overflow in the Residue Number System (RNS) is suggested. It works with the code whose redundancy is in the form of magnitude indices. The residue representation of a number with magnitude index is reconsidered. The RNS with magnitude index were first studied by Sasaki16 and Rao15 and then by Barsi and Maestrini15,6. The range of a given RNS is divided into intervals of equal width and the magnitude of a number X is defined as an integer locating X into one of such intervals. We have proposed algorithm which detects and corrects multiple errors in residue number. The algorithms for special cases viz., single burst residue error and single residue error are also suggested. Some of the advantages are pointed out over the existing approaches.

Introduction

Residue Number Systems (RNS) have been of interest to mathematicians for a very long time. However, the use of RNS to perform machine computation has attracted attention only since the late fifties (Aiken1, Garner2, Savobode3). Szabo and Tanaka18 have provided a comprehensive review of the work done in this field till the mid-sixties. Subsequent work was done by Bancroft3,4 and by Kaushik10,11,12. In addition, special applications of RNS have been investigated by other researchers.

The flurry of activity in the use of RNS for computation results from an inherent parallelism provided by residue arithmetic. A major advantage of using RNS in computer design is that it has an important capability of error detection and correction. The need to increase the degree of system reliability has led to the study of error correcting codes. Work in this direction has been done by Chien', Diamond8, Mandelbaum13,14, Rao15, Sasaki16, Barsi15,6 and Arora & Saroj12. The technology of fault-tolerant computing is, at the present time, in its infancy and much development is still required.

In this paper, a new approach for correction of multiple errors and detection of additive overflow in residue number has been suggested. This approach works with the code whose redundancy is in the form of magnitude indices. The range of a given RNS is divided into intervals of equal width and the magnitude of a number X is defined as an integer locating X into one of such intervals. Further, algorithms for special cases, viz., single burst residue error and single residue error are also developed. Superiority of the proposed algorithms over the existing ones, has been shown.

Residue Code With Magnitude Indices

Let \( m_1, m_2, \ldots, m_n \) be an ordered set of pairwise relatively prime positive integers such that \( m_i \nmid 2 \) for any \( 1 \leq i \leq n \). These \( m_i \)'s are called moduli. The corresponding ordered \( n \)-tuple \( (x_1, x_2, \ldots, x_n) \) of least non-negative residues of a number \( X \) with respect to the above moduli is called the residue representation of \( X \). Such a representation of numbers forms the RNS. Since all moduli are relatively prime, each \( X \in \mathbb{Z} \), where \( M = \prod_{i=1}^{n} m_i \), is uniquely represented in the RNS. Define an integer

\[
(x_i) = \lfloor X/m_i \rfloor, \quad \text{for any } 1 \leq i \leq n
\]

as a magnitude index of an integer \( X \) with respect to a modulus \( m_i \). It locates \( X \) into one interval of width \( m_i \). Represent \( X \) by \( (n+p) \)-tuple as

\[
X = (x_1, x_2, \ldots, x_n, (I_1^X, (I_2^X, \ldots, (I_p^X)),
\]

where \( 1 \leq p \leq n \),

which is called a redundant representation of \( X \) with magnitude indices. It is easy to see that \( (I_i^X) \in \{0, M_i\} \), where \( M_i = m_i/m_i \) and \( M \) can be computed from the residue digit of \( X \), using only two modular operations, i.e.,

\[
|I_i^X| M_i = \lfloor x_i/m_i \rfloor x_i m_i, \quad t = 1, 2, \ldots, i-1, \ldots, n.
\]
Therefore, \((I^f_X)\) can be represented uniquely with respect \(X\) to the moduli \(m_1, m_2, \ldots, m_n\) and it is represented as

\[
(I^f_X) = (I^f_X, i = 1, 2, \ldots, n) \quad (I^f_X, i = 1, 2, \ldots, n) = \left(\begin{array}{c}
I^f_X, 1, \ldots, I^f_X, N \end{array}\right).
\]

where \(I^f_X, i = |(I^f_X)| m_i, \quad i = 1, 2, \ldots, 1, i = 1, 1, \ldots, n\).

Additive properties of the residue representation with magnitude indices are straightforward.

Let \(X = (x_1, x_2, \ldots, x_n, (I^f_X, 1), (I^f_X, 2), \ldots, (I^f_X, p))\), and \(Y = (y_1, y_2, \ldots, y_n, (I^f_Y, 1), (I^f_Y, 2), \ldots, (I^f_Y, p))\) be two positive integers in the range \([0, M]\). Then

\[
X + Y = \left(\begin{array}{c}
|X + Y|_{m_1}, |X + Y|_{m_2}, \ldots, |X + Y|_{m_n}
\end{array}\right) \quad (I^f_{X+Y}) = (I^f_X + I^f_Y), \ldots, (I^f_{X+Y})
\]

where \(\delta_k = 1\) in case of addition, if \(x_k + y_k < m_k\) and \(\delta_k = 0\) in case of subtraction, if \(x_k - y_k < 0\), otherwise \(\delta_k = m_k\).

An additive overflow is detected if \((I^f_X)\) falls outside of \([0, M]\) for all \(i\), this situation can be overcome by adding a redundant modulus \(m_i + 1\) and consequently, representing \(X\) in the extended range \([0, M + 1]\), the extended range \([0, M + 1]\) contains all possible sums of \(X\) in the range \([0, M]\).

Let \(X = (x_1, x_2, \ldots, x_n, (I^f_X, 1), (I^f_X, 2), \ldots, (I^f_X, p))\), the (n+1)-tuple representing the integers in the range \([0, M + 1]\) are legitimate numbers. If any two legitimate numbers are added, then the sum may lie in the range \([M, 2M - 1]\), so we call it as the overflow subrange.

Multiple Error Correction in the Residue Number System with the Magnitude Indices

Let \(f\) be the number of errors occurring in the redundant representation of \(X \in [0, 2M - 1]\) with the magnitude indices.

Let \(X = (X_1, X_2, \ldots, X_n, (I^f_X, 1), (I^f_X, 2), \ldots, (I^f_X, p))\), be the redundant representation of an erroneous number \(X \in [0, M + 1]\) containing \(f\) errors in \(X_i\), where

\[
X = (x_1, x_2, \ldots, x_n, (I^f_X, 1), (I^f_X, 2), \ldots, (I^f_X, p))
\]

Compute the indices \((I^f_X, 1), (I^f_X, 2), \ldots, (I^f_X, p)\) with the help of the residue digits \(X_1, X_2, \ldots, X_n\).

Define \((D_i) = (I^f_X - (I^f_X, i), i = 1, 2, \ldots, p\).

We prove the following theorem.

Theorem 1: Given \(X \in [0, M + 1]\) to be an erroneous number which has \(f + 1\) errors among the

The left hand side of the above equation has two moduli common, say \(m_k\) and \(m_g\), then

\[
m_k \cdot m_g \left[ \frac{M \cdot m_{k+1}}{m_k \cdot m_g} - \beta \cdot \frac{m_k \cdot m_{g+1}}{m_k \cdot m_g} \right] = y, \quad q = 1, \ldots, f + 1, \quad \beta \cdot \frac{m_k \cdot m_{g+1}}{m_k \cdot m_g} = y_1 \text{(say)}
\]

Here \(y_1\) is a fraction because \(\min \{m_k \cdot m_g\} > m_k, 1 \leq f, g \leq n + 1\).
But the left hand side of the above equation can not be fraction for any value of \( x \). Hence this equation is inconsistent which implies that our assumption is not correct and thus \((D_i)\) has at most \( \lambda \) residue digits to be zero. Hence the theorem.

**Corollary**

In a system of \((n+1)\) moduli, where \(\min(m_1, m_2, ..., m_n) > m_n\), \(1 < f, g < n+1\), only \( \lambda \) residue digits error can be corrected if \( \lambda \leq (n-1)/2 \).

**Proof**

Let the residue digit \( x_i \) be in error and \( x_i \neq 0 \). Since \( x_i \) is an error digit, \((D_i)\) has at most \( \lambda \) residue digits to be zero (using theorem 1). As \( x_i \) is error free residue digit, \((D_j)\) has at least \((n-\lambda)\) residue digits to be zero. In order to distinguish \((D_i)\) from \((D_j)\), \((D_i)\) must have at least \((n+1)\) residue digits to be zero, i.e., \(4n+4\leq \lambda \leq (n-1)/2 \). Hence the corollary.

Consider the RRNS of \((n+1)\) moduli \( m_1, m_2, ..., m_n \) such that \(\min(m_1, m_2, ..., m_n) > m_f\).

1. If for some \( i \), \( m_i > m_f \) and \( 1 < f, g < n+1 \), let \((I_i^X, I_i^X)\) be the first \( \lambda \) indices in the redundant representation of \( X \).
2. Define \((D_i) = (I_i^X - I_i^X), i=1,2, ..., \lambda \).

**Theorem 2**

Under the assumption that no more than \( \lambda \) residue digits are in error, the following assertions are true.

**Assertion 1**

If \( \lambda \) of the \((D_i)\) = 0, then the residue digits corresponding to the moduli \( m_1, m_2, ..., m_n \) are correct and the errors lie among the indices. The errors are corrected by replacing the indices \((I_i^X, I_i^X), ..., (I_{n+1}^X, I_{n+1}^X)\) with the computed indices \((I_{n+1}^X, I_{n+1}^X), ..., (I_{n+1}^X, I_{n+1}^X)\).

**Assertion 2**

If \( \lambda \) of the \((D_i)\) have equal number of nonzero residue digits corresponding to the moduli \( m_{j_1}, m_{j_2}, ..., m_{j_k} \) where \( j_l = 1, 2, ..., 1 + 1, 1 + 1, ..., n + 1, t = 1, 2, ..., k \) and \( 1 \leq k \leq \lambda \) and at least \((\lambda-1)\) of the \((D_i)\) = 0, then the residue digits \( \bar{x}_{j_1}, \bar{x}_{j_2}, ..., \bar{x}_{j_k} \) are the only erroneous digits \( \bar{x}_{j_1}, \bar{x}_{j_2}, ..., \bar{x}_{j_k} \) are the only erroneous digits among the residues \( \bar{x}_1, \bar{x}_2, ..., \bar{x}_{n+1} \).

The erroneous digits are corrected as \( \bar{x}_{j_t} = | \bar{x}_{j_t} - |m_{j_1} - |m_{j_1}|., \bar{x}_{j_t} - |m_{j_1}|., \bar{x}_{j_t} - |m_{j_1}|., \bar{x}_{j_t} - |m_{j_1}|. t = 1, 2, ..., k \).

If \( k < \lambda \), then atmost \((\lambda-k)\) errors can occur among the indices and can be corrected by computing the new indices with the help of correct digits.

**Assertion 3**

If all the \((D_i) = 0\), then there is no error.

**Proof**

Let \( X = (x_1, x_2, ..., x_{n+1}, I_1^X, I_2^X, ..., I_{n+1}^X) \) be an erroneous number having atmost \( \lambda \) errors in \( X \), where \( x_1, x_2, ..., x_{n+1}, I_1^X, I_2^X, ..., I_{n+1}^X \) are the computed indices with the help of the residue digits \( \bar{x}_1, \bar{x}_2, ..., \bar{x}_{n+1} \) and form \((D_i) = (I_i^X - I_i^X), i = 1, 2, ..., \lambda \).

**Proof of Assertion 1**

On the contrary, assume that \( d \) of the residue digits among \( \bar{x}_1, \bar{x}_2, ..., \bar{x}_{n+1} \) are in error, where \( 0 < d < \lambda \). Then atmost \( d \) of the \((D_i)\) have nonzero residue digits to be zero (using theorem 1). In other words, atmost \( d \) of the \((D_i)\) have at least \((n-d)\) nonzero residue digits or atmost \( d \) of the \((D_i)\) have at least \( d \) nonzero residue digits, since \( d < n-d \). Also atmost \((n-d)\) of the \((D_i)\) have \( d \) nonzero residue digits. Hence \( \lambda \) of the \((D_i)\) have at least \( d \) nonzero residue digits. Since \( d \) of the errors occur among the residue digits, \( \bar{x}_1, \bar{x}_2, ..., \bar{x}_{n+1} \) therefore, atmost \( (\lambda-d) \) errors can occur among the indices.

Now \((D_i) = (I_i^X - I_i^X), i = 1, 2, ..., \lambda \).

**Assertion 2**

If \( \lambda \) of the \((D_i)\) have equal number of nonzero residue digits corresponding to the moduli \( m_{j_1}, m_{j_2}, ..., m_{j_k} \) where \( j_l = 1, 2, ..., 1 + 1, 1 + 1, ..., n + 1, t = 1, 2, ..., k \) and \( 1 \leq k \leq \lambda \), then the residue digits \( \bar{x}_{j_1}, \bar{x}_{j_2}, ..., \bar{x}_{j_k} \) are the only erroneous digits \( \bar{x}_{j_1}, \bar{x}_{j_2}, ..., \bar{x}_{j_k} \) are the only erroneous digits among the residues \( \bar{x}_1, \bar{x}_2, ..., \bar{x}_{n+1} \).

The erroneous digits are corrected as \( \bar{x}_{j_t} = | \bar{x}_{j_t} - |m_{j_1} - |m_{j_1}|., \bar{x}_{j_t} - |m_{j_1}|., \bar{x}_{j_t} - |m_{j_1}|., \bar{x}_{j_t} - |m_{j_1}|. t = 1, 2, ..., k \).

If \( k < \lambda \), then atmost \((\lambda-k)\) errors can occur among the indices and can be corrected by computing the new indices with the help of correct digits.

**Assertion 3**

If all the \((D_i) = 0\), then there is no error.

**Proof**

Let \( X = (x_1, x_2, ..., x_{n+1}, I_1^X, I_2^X, ..., I_{n+1}^X) \) be an erroneous number having atmost \( \lambda \) errors in \( X \), where \( x_1, x_2, ..., x_{n+1}, I_1^X, I_2^X, ..., I_{n+1}^X \) are the computed indices with the help of the residue digits \( \bar{x}_1, \bar{x}_2, ..., \bar{x}_{n+1} \) and form \((D_i) = (I_i^X - I_i^X), i = 1, 2, ..., \lambda \).
indices \((\bar{I}_1^1, \bar{I}_1^2, \ldots, \bar{I}_n^2')\).

Proof of Assertion 2

On the contrary, assume that the residue digits corresponding to the moduli \(m_{j_1}, m_{j_2}, \ldots, m_{j_k}\) are correct. Since all the residue digits corresponding to the moduli \(m_{j_1}, m_{j_2}, \ldots, m_{j_k}\) are not correct otherwise at least \(f\) of the \((\bar{B}_i)\) is \((0)\), which is contrary because at least \((2^f - 1)\) of the \((\bar{B}_i)\) are \((0)\), therefore, assume that \(h\) of the residue digits \(\bar{x}_{s_1}, \bar{x}_{s_2}, \ldots, \bar{x}_{s_h}\), where \(1 \leq s_u \leq n+1; 1 \leq u \leq h\) are in error. Then at least \(h\) of the \((B_i)\) have at least \((n-h)\) nonzero residue digits and at least \((2^f-h)\) of the \((\bar{B}_i)\) have equal number of nonzero residue digits corresponding to the moduli \(m_{s_1}, m_{s_2}, \ldots, m_{s_h}\).

Since at least \((2^f - h)\) of the errors can occur among the indices, therefore, at least \((2^f - h)\) of the \((B_i)\) are \((0)\). Hence at least \((f)\) of the \((\bar{B}_i)\) have equal number of nonzero residue digits corresponding to the moduli \(m_{s_1}, m_{s_2}, \ldots, m_{s_h}\), which is a contradiction since at least \((f)\) of the \((\bar{B}_i)\) have equal number of nonzero residue digits corresponding to the moduli \(m_{j_1}, m_{j_2}, \ldots, m_{j_k}\) and residue digits corresponding to the other moduli are zero. Therefore, our assumption that the residue digits corresponding to the moduli \(m_{j_1}, m_{j_2}, \ldots, m_{j_k}\) are correct is wrong. Now we show that \(\bar{x}_{j_1}, \bar{x}_{j_2}, \ldots, \bar{x}_{j_k}\) are the only erroneous digits among the residues \(\bar{x}_{1}, \bar{x}_{2}, \ldots, \bar{x}_{n+1}\).

If \(k \leq f\), then assume that \(\bar{x}_{s_h}\) is another erroneous residue digit. Then at least \((f)\) of the \((\bar{B}_i)\) have equal number of nonzero residue digits corresponding to the moduli \(m_{j_1}, m_{j_2}, \ldots, m_{j_k}\) and \(m_{s_h}\) which is again a contradiction. These errors can be corrected as follows:

\[
X = m_{j_1} \cdot (I_X^1) + x_1 \quad \text{and} \quad \bar{X} = m_{j_1} \cdot (I_X^1') + \bar{x}_{j_1}, \quad \text{where} \quad i \quad \text{corresponds to} \quad (\bar{B}_i), \quad \text{which has nonzero residue digits corresponding to the moduli} \quad m_{j_1}, m_{j_2}, \ldots, m_{j_k}.
\]

Then \(X = X_{m_{j_1} \cdot (I_X^1) + x_1} \quad \text{and} \quad \bar{X} = X_{m_{j_1} \cdot (I_X^1') + \bar{x}_{j_1}}, \quad \text{where} \quad \bar{x}_{j_1} \quad \text{corresponds to} \quad (\bar{B}_i)\).

\[
|X - X|_{m_{j_1} \cdot m_{j_2} \cdot \ldots \cdot m_{j_k}} = |m_{j_1} \cdot (B_i)|_{m_{j_2} \cdot \ldots \cdot m_{j_k}, t = 1, 2, \ldots, k},
\]

or

\[
x_{j_t} = \bar{x}_{j_t} - |m_{j_1} \cdot (B_i)|_{m_{j_2} \cdot \ldots \cdot m_{j_k}}.
\]

Since \((I_X^1)^{-1} = I_X\), for \(i\) as defined above, therefore, \(X = (B_i) = (\bar{B}_i)\) and hence

\[
x_{j_t} = |\bar{x}_{j_t} - |m_{j_1} \cdot (B_i)|_{m_{j_2} \cdot \ldots \cdot m_{j_k}}\|
\]

Then these residues are used to compute correct indices.

Proof of Assertion 3

Since all the \((\bar{B}_i) = (0)\), then the computed and the given indices are same and thus there is no error. Hence the theorem.

Now the following procedure is used to correct atmost \(f\) residue digit errors in the redundant representation of \(X\) with the magnitude indices.

Procedure 1

Step 1: Compute the indices \((\bar{I}_1^1), (\bar{I}_1^2), \ldots, (\bar{I}_1^2')\) with the help of residue digits \(\bar{x}_{1}, \bar{x}_{2}, \ldots, \bar{x}_{n+1}\).

Then from \(\bar{B}_i = (I_X^1) - (I_X^1'), i = 1, 2, \ldots, 2^f\).

Step 2: If all the \((\bar{B}_i) = (0)\), then there is no error and go to step 5.

Step 3: If at least \((f)\) of the \((\bar{B}_i) = (0)\), then the residue digits corresponding to the moduli \(m_{j_1}, m_{j_2}, \ldots, m_{j_k}\) and \(m_{s_h}\) are correct and errors lie among the other indices. The computed indices are the correct ones, go to step 5.

Step 4: If at least \((f)\) of the \((\bar{B}_i)\) have equal number of nonzero residue digits corresponding to the moduli \(m_{j_1}, m_{j_2}, \ldots, m_{j_k}\) and \(m_{s_h}\), then the residue digits \(\bar{x}_{j_1}, \bar{x}_{j_2}, \ldots, \bar{x}_{j_k}\) are in error and can be corrected as

\[
x_{j_t} = |\bar{x}_{j_t} - |m_{j_1} \cdot (B_i)|_{m_{j_2} \cdot \ldots \cdot m_{j_k}}\|, t = 1, 2, \ldots, k.
\]

Almost \((2^f - k)\) of the errors can lie among the indices which are corrected by computing the new indices with the help of correct residue digits.

Step 5: Use the residue digits corresponding to the moduli \(m_{j_1}, m_{j_2}, \ldots, m_{n+1}\) and compute \(n\)-residue \(X_{n+1}\) corresponding to the modulus \(n+1\) using the BEO. If \(X_{n+1} \neq \bar{X}_{n+1}\), then the number \(X\) is in overflow subrange; otherwise it is in the legitimate range. Stop.

Special Cases

(A) Burs Residue-Error Correction with Overflow Detection.

Assume that the set of \(N\)-r moduli \((m_1, m_2, \ldots, m_n)\) is in the ascending order.

Adjoin \(R\)-modulus \(m_{n+1} \geq 2\) to the \(N\)-r moduli in order to detect \(n+1\) as an additive overflow.
such that \( \min (m_1', m_0') > m_{f+1} \)

\[
l_1 \leq l \leq f + 1
\]

Consider the first \((f + 1)\) indices \((I_1^1), (I_2^2), \ldots, (I_{f+1}^{f+1})\) in the redundant representation of any \( x \in \mathbb{Z}[0, 2M - 1] \).

Let \( \tilde{x} = (\tilde{x}_1^1, \tilde{x}_2^2, \ldots, \tilde{x}_{f+1}^{f+1}) \) be an erroneous number \( \tilde{x} \in \mathbb{Z}[0, M, m_{f+1}] \) having a single burst residue error of length \( \leq f \) in the redundant representation of \( x \). Compute the indices \((I_1^1), (I_2^2), \ldots, (I_{f+1}^{f+1})\) with the help of the residue digits \( \tilde{x}_1^1, \tilde{x}_2^2, \ldots, \tilde{x}_{f+1}^{f+1} \).

Define \( (\tilde{B}_i^i) = ((I_1^1), (I_2^2), \ldots, (I_{f+1}^{f+1})) \), \( i = 1, 2, \ldots, f + 1 \).

**Theorem 3**

Assuming that no more than a single burst residue error of length less than or equal to \( f \) occurs, the following assertions are true.

**Assertion 1**

If one or two of the \((\tilde{B}_i^i)\) have at least \( f \) nonzero residue digits and the rest of the \((\tilde{B}_i^i) = (0)\), then the residue digits corresponding to the moduli \( m_1^1, m_2^2, \ldots, m_{f+1} \) are correct and \((I_1^1), (I_2^2), \ldots, (I_{f+1}^{f+1})\) are the erroneous indices. The errors are corrected by replacing \( (I_1^1) \) with the computed indices \((I_1^1) \).

**Assertion 2**

If \((\tilde{B}_2^2), (\tilde{B}_3^3), \ldots, (\tilde{B}_{f+1}^{f+1})\) have equal number of nonzero residue digits corresponding to the moduli \( m_{n-t+2}, m_{n-t+3}, \ldots, m_{n+1} \), for \( 0 < t < f \), and \((\tilde{B}_1^1)\) has at most \( (f-t) \) nonzero residue digits corresponding to the moduli \( m_2, m_3, \ldots, m_{f-t+1}, m_{f-t+1} \), then the residue digits \( \tilde{x}_1^{n-t+2}, \tilde{x}_2^{n-t+3}, \ldots, \tilde{x}_{n+1} \) and the first \( (f-t) \) residue digits of the index \((I_2^2)\) are erroneous. The errors are corrected as

\[
x_{n-t+k} = \tilde{x}_{n-t+k} - |m_j^j| \cdot (\tilde{B}_j^j) \cdot m_{n-t+k}
\]

for \( k = 2, 3, \ldots, f+1 \) and for some \( j, 2 \leq j \leq f+1 \).

The index \((I_1^1)\) is corrected by replacing it with residue corresponding to the moduli \( m_1^1, m_2^2, \ldots, m_{f+1} \).

**Assertion 3**

If at least one of \((\tilde{B}_j^j)\) has at most \( f \) nonzero residue digits and rest of the \((\tilde{B}_i^i)\) have at least \( (n-f) \) nonzero residue digits, then the indices \((I_1^1), (I_2^2), (I_{f+1}^{f+1})\) are correct. The errors lie among the residue digits \( x_1^{n-t+2}, x_2^{n-t+3}, \ldots, x_{n+1} \). The erroneous residue digits are corresponding to those moduli \( m_j^j \) whose residue digits in \((\tilde{B}_1^1)\) are nonzero. The errors are corrected as

\[
x_q = |\tilde{x}_q - |m_1^1| \cdot (\tilde{B}_1^1)| \cdot m_q\]

**Assertion 4**

If all the \((\tilde{B}_i^i) = (0)\), then there are no errors.

**Proof**

For the sake of brevity, we leave the proof. These assertions can be proved in the same manner as the assertions proved as earlier.

We have the following procedure for correcting a single burst residue error of length \( \leq f \).

**Procedure 2**

1. **Step 1:** Compute the indices \((I_2^2), (I_3^3), \ldots, (I_{f+1}^{f+1})\) with the help of the residue digits \( x_1^1, x_2^2, \ldots, x_{n+1} \). Then form \((\tilde{B}_i^i) = ((I_1^1), (I_2^2), \ldots, (I_{f+1}^{f+1}))\), \( i = 1, 2, \ldots, f+1 \).

2. **Step 2:** If all the \((\tilde{B}_i^i) = (0)\), then there are no errors and go to step 6.

3. **Step 3:** If at least one of \((\tilde{B}_i^i)\) has at most \( f \) nonzero residue digits and rest of the \((\tilde{B}_i^i)\) have at least \( (n-f) \) nonzero residue digits, then the indices are correct and go to step 7.

4. **Step 4:** If one or two of the \((\tilde{B}_i^i)\) have at most \( f \) nonzero residue digits and the rest of the \((\tilde{B}_i^i) = (0)\), then the residue digits corresponding to the moduli \( m_1^1, m_2^2, \ldots, m_{f+1} \) are correct. The errors lie among the residue digits of the indices \((I_1^1)\) and can be corrected by replacing with the computed indices \((I_1^1)\). Go to step 8.

5. **Step 5:** If all the \((\tilde{B}_i^i), i = 2, 3, \ldots, f+1\) have equal number of nonzero residue digits corresponding to the moduli \( m_{n-t+2}, m_{n-t+3}, \ldots, m_{n+1} \), for \( 0 < t < f \), and \((\tilde{B}_1^1)\) has at most \( (f-t) \) nonzero residue digits corresponding to the moduli \( m_2, m_3, \ldots, m_{f-t+1}, m_{f-t+1} \), then the residue digits \( \tilde{x}_1^{n-t+2}, \tilde{x}_2^{n-t+3}, \ldots, \tilde{x}_{n+1} \) and the first \( (f-t) \) residue digits of the index \((I_2^2)\) are erroneous. These errors are corrected as

\[
x_{n-t+k} = |\tilde{x}_{n-t+k} - |m_j^j| \cdot (\tilde{B}_j^j) \cdot m_{n-t+k}
\]

for \( k = 2, 3, \ldots, f+1 \) and for some \( j, 2 \leq j \leq f+1 \).

The index \((I_1^1)\) is corrected by replacing it with residue corresponding to the moduli \( m_1^1, m_2^2, \ldots, m_{f+1} \).

6. **Step 6:** These errors are corrected as

\[
x_{n-t+k} = |\tilde{x}_{n-t+k} - |m_j^j| \cdot (\tilde{B}_j^j) \cdot m_{n-t+k}\]

for \( k = 2, 3, \ldots, f+1 \) and for some \( j, 2 \leq j \leq f+1 \).

Compute a new index of the modulus \( m_j \) and replace the erroneous index \((I_1^1)\) with it. Go to step 8.
Step 7: The errors lie among the residue digits $x_1, x_2, \ldots, x_{n+1}$. These erroneous residue digits are corrected as

$$x_q = |\bar{x}_q - m_q \cdot (B_q)| \mod m_q$$

where $m_q$ are those moduli whose corresponding residue digits in $(B_q)$ are nonzero.

Step 8: Use $x_1, x_2, \ldots, x_n$ residue digits to compute $x_i$ corresponding to $R$-modulus $m_{n+1}$ by using the BEO. If $x_{n+1} \neq x_{n+1}$, then an overflow is detected; otherwise the number $X$ is in the legitimate range. Stop.

(B) Single Residue Digit-Error with Overflow:

For correcting single residue error, we represent $x \in \mathbb{Z}/(2^m \cdot m_{n+1})$ with first two magnitude indices $(1, \bar{1})$ as $x = (x_1, x_2, \ldots, x_{n+1}, (1)\bar{1})$.

Let $\bar{x} \leftrightarrow (x_1, x_2, \ldots, x_{n+1}, (1)\bar{1})$ be a redundant representation of an erroneous number $x \in \mathbb{Z}/(2^m \cdot m_{n+1})$ having only single residue digit error. Compute the indices $(1\bar{1})$ and $(1)\bar{1}$ with the help of the redundants $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}$ and define $B_q = ((1)\bar{1}) - (1)\bar{1}), \; q = 1, 2$.

Theorem 4

Assuming that only one residue digit error occurs the following assertions hold true provided $\min(m_x, m_g) \geq m_2$.

Assertion 1

If only one $(B_q) \neq (0)$, then the residue digits $x_1, x_2, \ldots, x_{n+1}$ are correct and the error lies in the index $(1\bar{1})$. The error is corrected by replacing $(1\bar{1})$ with the computed index $(1)\bar{1}$.

Assertion 2

If at least one of $(B_q)$ has only one nonzero residue digit corresponding to a modulus, say $m_q$, a $\bar{x}_{k+1}$, and rest of $(B_q)$ have least $(n+1)$ nonzero residue digits, then the indices are correct and $x_k$ is an erroneous residue digit. It is corrected as

$$x_k = |\bar{x}_k - m_1 \cdot (B_k)| \mod m_k$$

Assertion 3

If all the $(B_q) = (0)$, then there is no error.

Proof

These assertions can also be proved in the same manner.

By using these assertions, we can correct a single residue error.

Remark

This method can correct anyodd errors such that $\binom{n+1}{2} < f < n$, provided if we add $(2^m \cdot n)$ R-modulus to the given set of the moduli and consider only first $2^\ell$ indices.

Comparison

We have proposed a new approach for correcting multiple errors along with the detection of an additive overflow in residue code. This approach results in faster decoding procedures and has following advantages over those given by Mendelbaum, Barsi & Maestrini, and Arora & Saroj.

1. The procedures for single, burst, and multiple residue error correction based upon the proposed approach operate on the residue representation of the numbers and make exclusive use of the modular operations in contrast to the procedures suggested by Barsi & Maestrini, Mendelbaum, which require a separate positional processor for implementation.

2. Barsi & Maestrini has suggested another algorithm for single error detection & correction with magnitude index whereas the proposed approach is general for $f(n)$ number of errors. The definition of magnitude index in this paper is different from the one introduced by Barsi & Maestrini. This definition leads to faster decoding algorithms as compared to the one suggested by Barsi & Maestrini at the cost of redundancy. But in an era of VLSI technology, it is always desirable to have faster decoding algorithms.

3. If $\frac{n+1}{2} < f < n$, then the proposed approach needs $(2^m \cdot n+1)$ R-modulus whereas the method suggested by Arora & Saroj requires at least $(2^m \cdot 1)$ R-moduli.

The proposed procedures are faster as compared to those suggested by Arora & Saroj.

Table I shows the number of modular operations required by the proposed procedures $(P)$ and the procedures $(G)$ suggested earlier by author in [2]. Thus we see from the table that the proposed approach results in the faster decoding procedures.

Conclusion

In this paper, a new approach for correcting multiple errors along with detection of an additive overflow in residue code is presented. This approach works with the code whose redundancy is in the form of magnitude indices. Further,
<table>
<thead>
<tr>
<th>Type of Errors</th>
<th>Procedure (Q)</th>
<th>Number of Modular Operations Required</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiple</td>
<td>n+I/2</td>
<td>(2n+3r-2)+(2n+3r-1)</td>
</tr>
<tr>
<td></td>
<td>n+1</td>
<td></td>
</tr>
<tr>
<td>Burst</td>
<td>2n+5f+4</td>
<td></td>
</tr>
<tr>
<td>Single</td>
<td>[(n+I)/2]**(2n+2r+1)+(4n+6r-3)</td>
<td></td>
</tr>
</tbody>
</table>

procedures for special cases, viz., single burst residue error and single residue error are also suggested. The advantages of this approach have been stated over those of Arora & Saroj2, Barsi & Maestrini3,4 and Mandelbaum14.

References


* [I] denotes the floor of I i.e. the largest integer \( \leq I \).
** [I] denotes the ceiling of I i.e. the smallest integer \( \geq I \).


