# MULTIPLE ERROR CORRECTION AND ADDITIVE OVERFLOW DETECTION WITH MAGNITUDE INDICES IN RESIDUE CODE

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ABSTRACT -

A new approach for correcting multiple errors and detecting an additive overflow in the Residue Number System (RNS) is suggested. It works with the code whose redundancy is in the form of magnitude indices. The residue representation of a number with magnitude index is reconsidered. The RNS with magnitude index were first studied by Sasaki[16] and Rao[15] and then by Barsi and Maestrini[5,6]. The range of a given RNS is divided into intervals of equal width and the magnitude of a number X is defined as a integer locating X into one of such intervals. We have proposed algorithm which detects and corrects multiple errors in residue number. The algorithms for special cases viz., single burst residue error and single residue error are also suggested. Some of the advantages are pointed out over the existing approaches.

## Introduction

Residue Number Systems (RNS) have been of interest to mathematicians for a very long time. However, the use of RNS to perform machine computation has attracted attention only since the late fifties [Aiken[1], Garner[9], Savobode[17]]. Szabo and Tanaka[18] have provided a comprehensive review of the work done in this field till the mid-sixties. Subsequent work was done by Banerji[3,4] and by Kaushik[10,11,12]. In addition, special applications of RNS have been investigated by other researchers.

The flurry of activity in the use of RNS for computation results from an inherent parallelism provided by residue arithmetic. A major advantage of using RNS in computer design is that it has an important capability of error detection and correction. The need to increase the degree of system reliability has led to the study of error correcting codes. Work in this direction has been done by Chien[7], Diamond[8], Mandelbaum[13,14], Rao[15], Sasaki[16], Barsi[5,6] and Arora & Saroj[2]. The technology of fault-tolerant computing is, at the present time, in its infancy and much development is still required.

In this paper, a new approach for correction of multiple errors and detection of additive overflow in residue number has been suggested. This approach works with the code whose reundancy is in the form of magnitude indices. The range of a given RNS is divided into intervals of equal width and the magnitude of a number X is defined as an integer locating X into one of such intervals. Further, algorithms for special cases, viz., single burst reside error and single residue error are also developed. Superiority of the proposed algorithms over the existing ones, has been shown.

## Residue Code With Magnitude Indices

Let $(m_1, m_2, \ldots, m_n)$ be an ordered set of n pairwise relatively prime positive integers such that $m_i \geq 2$ for any $1 \leq i \leq n$. These mi's are called moduli. The corresponding ordered n-tuple $(x_1, x_2, \ldots, x_n)$ of least non-negative residues of a number X with respect to the above moduli is called the residue representation of X. Such a representation of numbers forms the RNS. Since all moduli are relatively prime, each $X \varepsilon [0, M)$, where $M = \prod_{i=1}^{n} m_i$, is uniquely represented in the RNS. Define an integer

$$(I_X^i) = \lfloor X/m_i \rfloor \text{ , for any } i, 1 \leq i \leq n$$

as a magnitude index of an integer X with respect to a modulus $m_i$. It locates X into one interval of width $m_i$. Represent X by (n+p)-tuple as

$$X \leftrightarrow (x_1, x_2, \ldots, x_n, (I_X^1), (I_X^2), \ldots, (I_X^p)),$$
$$1 \leq p \leq n,$$

which is called a redundant representation of X with magnitude indices. It is easy to see that $(I_X^i) \varepsilon [0, \hat{m}_i)$, where $\hat{m}_i = M/m_i$ and $(I_X^i)$ can be computed from the residue digits of X, using only two modular operations, i.e.,

$$|(I_X^i)|_{m_t} = \left|\left|\frac{1}{m_i}\right|_{m_t} |x_t - x_i|_{m_t}\right|_{m_t}, \; t = 1, 2, \ldots i-1, i+1 \ldots n.$$



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Therefore, $(I_X^i)$ can be represented uniquely with respect to the moduli $m_1, m_2, \ldots, m_{i-1}, m_{i+1} \ldots m_n$ and it is represented as

$$(I_X^i) \leftrightarrow (I_{X,1}^i, I_{X,2}^i, \ldots, I_{X,i-1}^i, I_{X,i+1}^i, \ldots I_{X,n}^i),$$

where $I_{X,t}^i = |(I_X^i)|_{m_t}$, $t=1,2,\ldots,i-1,i+1,\ldots n$.

Additive properties of the residue representation with magnitude indices are straight forward.

Let $X \leftrightarrow (x_1, x_2, \ldots, x_n, (I_X^1), (I_X^2), \ldots, (I_X^p))$;

and $Y \leftrightarrow (y_1, y_2, \ldots, y_n, (I_Y^1), (I_Y^2), \ldots, (I_Y^p))$

be two positive integers in the range $[0,M)$. Then

$$X+Y \leftrightarrow (|x_1 \pm y_1|_{m_1}, |x_2 \pm y_2|_{m_2}, \ldots, |x_n \pm y_n|_{m_n}$$
$$(|I_X^1 \pm I_Y^1 + d_1|_{\hat{m}_1}), (|I_X^2 \pm I_Y^2 + d_2|_{\hat{m}_2}), \ldots,$$
$$(|I_X^p \pm I_Y^p + d_p|_{\hat{m}_p})),$$

where $d_k=1$ in case of addition, if $x_k+y_k \geq m_k$ and $d_k=-1$ in case of subtraction, if $x_k-y_k<0$, otherwise $d_k=0$.

An additive overflow is detected if $(I_X^i)$ falls outside of $[0,\hat{m}_i)$ for all $i$. This situation can be overcome by adding a redundant modulus $m_{n+1} \geq 1$ and consequently, representing $X$ in the extended range $[0, M.m_{n+1})$. In this case, the indices $(I_X^i)$ and represented in the extended range $[0, \hat{m}_i.m_{n+1})$. The $(n+1)$-tuple representing the integers in the range $[0,M)$ are legitimate numbers. If any two legitimate numbers are added, then the sum may lie in the $[M, 2M-1)$ subrange. So we call it as the overflow subrange.

## Multiple Error Correction in the Residue Number System with the Magnitude Indices

Let $\ell$ be the number of errors occuring in the redundant representation of $X \varepsilon [0, 2M-1)$ with the magnitude indices.

Let $X \leftrightarrow (\bar{x}_1, \bar{x}_2, \ldots, x_{n+1}, (I_X^{\bar{1}}), (I_X^{\bar{2}}), \ldots, (I_X^{\bar{p}}))$.

be the redundant representation of an erroneous number $\bar{X} \varepsilon [0, M.m_{n+1})$ having almost $\ell$ errors in $X$, where

$$X \leftrightarrow (x_1, x_2, \ldots, x_{n+1}, (I_X^1), (I_X^2) \ldots (I_X^p))$$

Compute the indices $(I_{\bar{X}}^1), (I_{\bar{X}}^2), \ldots, (I_{\bar{X}}^p)$ with the help of the residue digits $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_{n+1}$.

Define $(D_i) = ((I_{\bar{X}}^i) - (I_X^i)), i=1,2,\ldots,p$.

We prove the following theorem.

**Theorem 1:**

Given $\bar{X} \varepsilon [0, M.m_{n+1})$ to be an erroneous number which has $\ell_1 (\leq \ell)$ errors among the residue digits of $X \epsilon [0, 2M-1)$. If $m_i$, $1 \leq i \leq n$ is one of the $\ell_1$ moduli whose corresponding residue digit $x_i$ is in error, then $(D_i)$ has at most $\ell_1$ residue digits to be zero, provided that $\ell_1 \leq n-1$ and $\text{Min.}(m_f.m_g) > m_n$, $1 \leq f,g \leq n+1$.

Proof:

On the contrary, assume that $(D_i)$ has at least $(\ell_1+1)$ residue digits to be zero corresponding to the moduli, say $m_{k_1}, m_{k_2}, \ldots m_{k_{\ell_1+1}}$ $1 \leq k_j \leq n+1$; $k_j \neq i$ and $j=1,2,\ldots,\ell_1+1$. Therefore, $(D_i) = \beta . \prod_{j=1}^{\ell_1+1} m_{k_j}$,

for some $\beta$, $0 \leq \beta < \dfrac{M.m_{n+1}}{\prod_{j=1}^{\ell_1+1} m_{k_j}}$

Let $m_{t_1}, m_{t_2}, \ldots, m_{t_{\ell_1}}$, $1 \leq t_q \leq n+1$; $1 \leq q \leq \ell_1$ be those moduli whose corresponding residue digits in $X$ are erroneous, then

$$\bar{X} = X + \nu . \frac{M.m_{n+1}}{\prod_{q=1}^{\ell_1} m_{t_q}}, \text{ for some } \nu, 0 \leq \nu < \prod_{q=1}^{\ell_1} m_{t_q}$$

Now $(I_X^i) = \lfloor X/m_i \rfloor$ and $(I_{\bar{X}}^i) = \lfloor \bar{X}/m_i \rfloor$.

Also $(D_i) = ((I_{\bar{X}}^i) - (I_X^i))$.

Therefore, $(I_{\bar{X}}^i) = (I_X^i) + \beta . \prod_{j=1}^{\ell_1+1} m_{k_j}$, or

$$\lfloor \bar{X}/m_i \rfloor = \lfloor X/m_i \rfloor + \beta . \prod_{j=1}^{\ell_1+1} m_{k_j}.$$

or

$$\left\lfloor \left(X + \nu . \frac{M.m_{n+1}}{\prod_{q=1}^{\ell_1} m_{t_q}}\right) / m_i \right\rfloor = \lfloor X/m_i \rfloor + \beta . \prod_{j=1}^{\ell_1+1} m_{k_j},$$

i.e. $\nu . \dfrac{M.m_{n+1}}{\prod_{q=1}^{\ell_1} m_{t_q}} - \beta . \prod_{j=1}^{\ell_1+1} m_{k_j} . m_i = y < m_i$

The left hand side of the above equation has two moduli common, say $m_f$ and $m_g$. Then

$$m_f.m_g \left[ \nu . \frac{M.m_{n+1}}{\prod_{q=1}^{\ell_1} m_{t_q}.m_f m_g} - \beta . \frac{\prod_{j=1}^{\ell_1+1} m_{k_j}.m_i}{m_f.m_g} \right] = y,$$

or

$$\nu . \frac{M.m_{n+1}}{\prod_{q=1}^{\ell_1} m_{t_q}.m_f.m_g} - \beta . \frac{\prod_{j=1}^{\ell_1+1} m_{k_j}.m_i}{m_f.m_g} = y_1 \text{ (say)}$$

Here $y_1$ is a fraction because $\text{Min.}(m_f.m_g) > m_n$, $1 \leq f,g \leq n+1$.

But the left hand side of the above equation can not be fraction for any value of and $\beta$. Hence this equation is inconsistent which implies that our assumption is not correct and thus $(D_i)$ has atmost $\lambda_1$ residue digits to be zero. Hence the theorem.

### Corollary

In a system of (n+1) moduli, where $\text{Min.}(m_f.m_g) > m_n$, $1 \le f,g \le n+1$, only $\lambda$ residue digits error can be corrected if $\lambda \le (n-1)/2$.

### Proof

Let the residue digit $x_i$ be in error and $x_j$ be an error free digit. Since $x_i$ is an erroneous digit, $(D_i)$ has atmost $\lambda$ residue digits to be zero (using theorem 1). As $x_j$ is error free residue digit, $(D_j)$ has atleast $(n-\lambda)$ residue digits to be zero. In order to distinguish $(D_j)$ from $(D_i)$, $(D_j)$ must have atleast $(\lambda+1)$ residue digits to be zero, i.e. $\lambda+1 \le n-\lambda \Rightarrow \lambda \le (n-1)/2$. Hence the corrollary.

Consider the RRNS of (n+1) moduli $m_1, m_2, \ldots, m_{n+1}$ such that $\text{Min.}(m_f.m_g) > m_{2\lambda}$; $1<f,g<n+1$. Let $(I_X^1), (I_X^2), \ldots (I_X^{2\lambda})$ be the first $2\lambda$ indices in the redundant representation of X.

Define $(\bar{D}_i) = ((I_{\bar{X}}^i)\ (\overline{I_X^i})), i=1,2,..2\lambda$.

### Theorem 2

Under the assumption that no more than $\lambda$ residue digits are in error, the following assertions are true.

### Assertion 1

If atleast $\lambda$ of the $(\bar{D}_i)=(0)$, then the residue digits corresponding to the moduli $m_1, m_2, \ldots m_{n+1}$ are correct and the errors lie among the indices. The errors are corrected by replacing the indices $(\overline{I_X^1}), (\overline{I_X^2}), \ldots, (\overline{I_X^{2\lambda}})$ with the computed indices $(I_{\bar{X}}^1), (I_{\bar{X}}^2), \ldots, (I_{\bar{X}}^{2\lambda})$.

### Assertion 2

If atleast $\lambda$ of the $(\bar{D}_i)$ have equal number of nonzero residue digits corresponding to the moduli $m_{j_1}, m_{j_2}, \ldots, m_{j_k}$, where $j_t=1,2,\ldots,i-1,i+1,\ldots,n+1; t=1,2,\ldots k$ and $1 \le k \le \lambda$, and atmost $(\lambda-1)$ of the $(\bar{D}_i)=(0)$, then the residue digits $\bar{x}_{j_1}, \bar{x}_{j_2}, \ldots, \bar{x}_{j_k}$ are the only erroneous digits $\bar{x}_{j_1}, \bar{x}_{j_2}, \ldots, \bar{x}_{j_k}$ are the only erroneous digits among the residues $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}$. The erroneous digits are corrected as

$$x_{j_t} = \left| \bar{x}_{j_t} - |m_i.(\bar{D}_i)|_{m_{j_t}} \right|_{m_{j_t}}, \quad t=1,2,\ldots,k.$$

If $k < \lambda$, then atmost $(\lambda-k)$ errors can occur among the indices and can be corrected by computing the new indices with the help of correct digits.

### Assertion 3

If all the $(\bar{D}_i)=(0)$, then there is no error.

### Proof

Let $\bar{X} \leftrightarrow (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}, (\overline{I_X^1}), (\overline{I_X^2}), \ldots \ldots (\overline{I_X^{2\lambda}}))$ be an erroneous number having atmost $\lambda$ errors in $\bar{X}$, where

$X \leftrightarrow (x_1, x_2, \ldots, x_{n+1}, (I_X^1), (I_X^2), \ldots, (I_X^{2\lambda}))$.

Compute the indices $(I_{\bar{X}}^1), (I_{\bar{X}}^2), \ldots, (I_{\bar{X}}^{2\lambda})$ with the help of the residue digits $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}$ and form

$(\bar{D}_i) = ((I_{\bar{X}}^i) - (\overline{I_X^i}))$, $i=1,2,\ldots,2\lambda$.

### Proof of Assertion 1

On the contrary, assume that d of the residue digits among $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}$ are in error, where $0 < d < \lambda$. Then atmost d of the $(D_i)$ have atmost d residue digits to be zero (using theorem 1). In other words, atmost d of the $(D_i)$ have atleast (n-d) non-zero residue digits or atmost d of the $(D_i)$ have atleast d nonzero residue digits, since $d < n-d$. Also atleast $(2\lambda-d)$ of the $(D_i)$ have d nonzero residue digits. Hence $2\lambda$ of $(D_i)$ have atleast d nonzero residue digits. Since d of the errors occur among the residue digits, $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}$, therefore, atmost $(\lambda-d)$ errors can occur among the indices.

Now $(\bar{D}_i) = ((I_{\bar{X}}^i) - (\overline{I_X^i}))$; or

$(\bar{D}_i) = ((I_{\bar{X}}^i) - (I_X^i) + (I_X^i) - (\overline{I_X^i}))$; or

$(\bar{D}_i) = ((D_i) + (\bar{\bar{D}}_i))$, where $(\bar{\bar{D}}_i) = ((I_X^i) - (\overline{I_X^i}))$.

Hence atmost $(\lambda-d)$ of the $(D_i) \ne (\bar{D}_i)$ and atleast $(\lambda+d)$ of the $(D_i) = (\bar{D}_i)$. Therefore, atleast $(\lambda+d)$ of the $(\bar{D}_i)$ have atleast d nonzero residue digits and atmost $(\frac{\lambda-d}{d})$ of the $(\bar{D}_i)=(0)$, which is a contradiction of the given fact that atleast $\lambda$ of the $(\bar{D}_i)=(0)$. Hence the residue digits corresponding to the moduli $m_1, m_2, \ldots, m_{n+1}$ are correct and the errors lie among the indices. The errors are corrected by replacing the indices $(\overline{I_X^1}), (\overline{I_X^2}), \ldots (\overline{I_X^{2\lambda}})$ with the computed indices $(I_{\bar{X}}^{1}), (I_{\bar{X}}^{2}), \ldots (I_{\bar{X}}^{2\ell})$.

Proof of Assertion 2

On the contrary, assume that the residue digits corresponding to the moduli $m_{j_1}, m_{j_2}, \ldots, m_{j_k}$ are correct. Since all the residue digits corresesppnding to the moduli $m_1, m_2, \ldots, m_{n+1}$ are not correct otherwise atleast $\ell$ of the $(\bar{D}_i)=(0)$, which is contrary because atmost $(\ell-1)$ of the $(\bar{D}_i)=(0)$, therefore, assume that h of the residue digits $\bar{x}_{s_1}, \bar{x}_{s_2}, \ldots, \bar{x}_{s_h}$, where $1 \leq s_u \leq n+1$; $1 \leq u \leq h$ are in error. Then atmost $\bar{h}$ of the $(\bar{D}_i)$ have atleast (n-h) nonzero residue digits and atleast $(2\ell-h)$ of the $(D_i)$ have equal number of nonzero residue digits corresponding to the moduli $m_{s_1}, m_{s_2}, \ldots, m_{s_h}$.

Since atmost $(\ell-h)$ of the errors can occur among the indices, therefore, atmost $(\ell-h)$ of the $(D_i) \neq (\bar{D}_i)$. Hence atleast $\ell$ of the $(\bar{D}_i)$ have equal number of nonzero residue digits corresponding to the moduli $m_{s_1}, m_{s_2}, \ldots, m_{s_h}$, which is a contradiction since atleast $\ell$ of the $(\bar{D}_i)$ have equal number of nonzero residue digits corresponding to the moduli $m_{j_1}, m_{j_2}, \ldots, m_{j_k}$ and residue digits corresponding to the other moduli are zero. Therefore, our assumption that the residue digits corresponding to the moduli $m_{j_1}, m_{j_2}, \ldots, m_{j_k}$ are correct, is wrong. Now we show that $\bar{x}_{j_1}, \bar{x}_{j_2}, \ldots, \bar{x}_{j_k}$ are the only erroneous digits among the residues $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}$. If $k < \ell$, then assume that $\bar{x}_w$ is another erroneous residue digit. Then atleast $\ell$ of the $(\bar{D}_i)$ have equal number of nonzero residue digits corresponding to the moduli $m_{j_1}, m_{j_2}, \ldots, m_{j_k}, m_w$ which is again a contradiction. These errors can be corrected as follows:

$X = m_i \cdot (I_X^{i}) + x_i$ and $\bar{X} = m_i \cdot (I_{\bar{X}}^{i}) + \bar{x}_i$, where i corresponds to that $(\bar{D}_i)$ which has nonzero residue digits corresponding to the moduli $m_{j_1}, m_{j_2}, \ldots, m_{j_k}$.

Then $\bar{X} - X = m_i \cdot ((I_{\bar{X}}^{i}) - (I_X^{i})) + (\bar{x}_i - x_i)$, or

$\bar{X} - X = m_i \cdot (D_i)$, since $\bar{x}_i = x_i$.

Now $|\bar{X} - X|_{m_{j_t}} = |m_i \cdot (D_i)|_{m_{j_t}}$, $t = 1, 2, \ldots, k$,

or

$$x_{j_t} = \left| \bar{x}_{j_t} - |m_i \cdot (D_i)|_{m_{j_t}} \right|_{m_{j_t}}$$

Since $(I_X^{i}) = (\bar{I}_X^{i})$, for i as defined above, therefore, $(D_i) = (\bar{D}_i)$ and hence

$$x_{j_t} = \left| \bar{x}_{j_t} - |m_i \cdot (\bar{D}_i)|_{m_{j_t}} \right|_{m_{j_t}}$$

Then these residues are used to compute correct indices.

Proof of Assertion 3

Since all the $(\bar{D}_i)=(0)$, then the computed and the given indices are same and thus there is no error. Hence the theorem.

Now the following procedure is used to correct atmost $\ell$ residue digit errors in the redundant representation of X with the magnitude indices.

Procedure 1

Step 1: Compute the indices $(I_{\bar{X}}^{1}), (I_{\bar{X}}^{2}), \ldots, (I_{\bar{X}}^{2\ell})$ with the help of residue digits $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}$. Then from $(\bar{D}_i) = ((I_{\bar{X}}^{i}) - (\bar{I}_X^{i}))$, $i = 1, 2, \ldots, 2\ell$.

Step 2: If all the $(\bar{D}_i)=(0)$, then there is no error and go to step 5.

Step 3: If atleast $\ell$ of the $(\bar{D}_i)=(0)$, then the residue digits corresponding to the moduli $m_1, m_2, \ldots, m_{n+1}$ are correct and errors lie among the indices. The computed indices are the correct ones, go to step 5.

Step 4: If atleast $\ell$ of the $(\bar{D}_i)$ have equal number of nonzero residue digits corresponding to the moduli $m_{j_1}, m_{j_2}, \ldots, m_{j_k}$, $(k \leq \ell)$, then the residue digits $\bar{x}_{j_1}, \bar{x}_{j_2}, \ldots, \bar{x}_{j_k}$ are in error and can be corrected as

$$x_{j_t} = \left| \bar{x}_{j_t} - |m_i \cdot (\bar{D}_i)|_{m_{j_t}} \right|_{m_{j_t}}, \quad t = 1, 2, \ldots, k.$$

Atmost $(\ell-k)$ of the errors can lie among the indices which are corrected by computing the new indices with the help of correct residue digits.

Step 5: Use the residue digits corresponding to the moduli $m_1, m_2, \ldots, m_{n+1}$ and compute R-residue $x'_{n+1}$ corresponding to the modulus $m_{n+1}$ using the BEO. If $x'_{n+1} \neq \bar{x}_{n+1}$, then the number X is in overflow subrange; otherwise it is in the legitimate range. Stop.

## Special Cases

(A) Burst Residue-Error Correction with Overflow Detection.

Assume that the set of N-R moduli $(m_1, m_2, \ldots, m_n)$ is in the ascending order. Adjoin R-modulus $m_{n+1} \geq 2$ to the N-R moduli in order to detect an additive overflow such that $\mathrm{Min.}(m_f, m_g) > m_{\ell+1}$

$$1 \leq f, g \leq n+1$$

Consider the first $(\ell+1)$ indices $(I_X^1), (I_X^2), \ldots, (I_X^{\ell+1})$ in the redundant representation of any $X \varepsilon [0, 2M-1)$.

Let $\bar{X} \leftrightarrow (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}, (\overline{I_X^1}), (\overline{I_X^2}), \ldots, (\overline{I_X^{\ell+1}})$ be an erroneous number $\bar{X} \varepsilon [0, M.m_{n+1})$ having a single burst residue error of length $\leq \ell$ in the redundant representation of $\bar{X}$. Compute the indices $(I_{\bar{X}}^1), (I_{\bar{X}}^2), \ldots, (I_{\bar{X}}^{\ell+1})$ with the help of the residue digits $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}$.

Define $(\bar{D}_i) = ((I_{\bar{X}}^i) - (\overline{I_X^i})), i=1, 2, \ldots \ell+1.$

**Theorem 3**

Assuming that no more than a single burst residue error of length less than or equal to $\ell$ occurs, the following assertions are true.

**Assertion 1**

If one or two of the $(\bar{D}_i)$ have atmost $\ell$ nonzero residue digits and the rest of the $(\bar{D}_i) = (0)$, then the residue digits corresponding to the moduli $m_1, m_2, \ldots, m_{n+1}$ are correct and $(\overline{I_X^i})$ are the erroneous indices. The errors are corrected by replacing $(\overline{I_X^i})$ with the computed indices $(I_{\bar{X}}^i)$.

**Assertion 2**

If $(\bar{D}_2), (\bar{D}_3), \ldots, (\bar{D}_{\ell+1})$ have equal number of nonzero residue digits corresponding to the moduli $m_{n-t+2}, m_{n-t+3}, \ldots, m_{n+1}$, for $0 < t < \ell$ and $(\bar{D}_1)$ has atmost $(\ell-t)$ nonzero residue digits corresponding to the moduli $m_2, m_3, \ldots, m_{\ell-t}, m_{\ell-t+1}$, then the residue digits $\bar{x}_{n-t+2}, \bar{x}_{n-t+3} \ldots \bar{x}_{n+1}$ and the first $(\ell-t)$ residue digits of the index $(\overline{I_X^1})$ are erroneous. The errors are corrected as

$$x_{n-t+k} = \left| \bar{x}_{n-t+k} - \left| m_j . (\bar{D}_j) \right|_{m_{n-t+k}} \right|_{m_{n-t+k}},$$

$k=2, 3, \ldots, t+1$ and for some $j, 2 \leq j \leq \ell+1$. The index $(\overline{I_X^1})$ is corrected by replacing it with residues corresponding to the moduli $m_1, m_2, \ldots, m_{n+1}$.

**Assertion 3**

If atleast one of $(\bar{D}_i)$ has atmost $\ell$ nonzero residue digits and rest of the $(\bar{D}_i)$ have atleast $(n-\ell)$ nonzero residue digits, then the indices $(\overline{I_X^1}), (\overline{I_X^2}), (\overline{I_X^{\ell+1}})$ are correct. The errors lie among the residue digits $\bar{x}_1, \bar{x}_2, \ldots, x_{n+1}$. The erroneous residue digits are corresponding to those moduli $m_q$ whose residue digits in $(\bar{D}_i)$ are nonzero. The errors are corrected

$$x_q = \left| \bar{x}_q - \left| m_i . (\bar{D}_i) \right|_{m_q} \right|_{m_q}.$$

**Assertion 4**

If all the $(\bar{D}_i) = (0)$, then there are no errors.

**Proof**

For the sake of brevity, we leave the proof. These assertions can be proved in the same manner as the assertions proved as earlier.

We have the following procedure for correcting a single burst residue error of length $\leq \ell$.

**Procedure 2**

**Step 1:** Compute the indices $(I_{\bar{X}}^1), (I_{\bar{X}}^2), \ldots (I_{\bar{X}}^{\ell+1})$ with the help of the residue digits $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}$. Then form

$(\bar{D}_i) = ((I_{\bar{X}}^i) - (\overline{I_X^i})), i=1, 2, \ldots, \ell+1.$

**Step 2:** If all the $(\bar{D}_i) = (0)$, then there are no errors and go to step 8.

**Step 3:** If atleast one of the $(\bar{D}_i)$ has atmost $\ell$ nonzero residue digits and rest of the $(\bar{D}_i)$ have atleast $(n-\ell)$ nonzero residue digits, then the indices are correct and go to step 7.

**Step 4:** If one or two of the $(\bar{D}_i)$ have atmost $\ell$ nonzero residue digits and the rest of the $(\bar{D}_i) = (0)$, then the residue digits corresponding to the moduli $m_1, m_2, \ldots, m_{n+1}$ are correct. The errors lie among the residue digits of the indices $(\overline{I_X^i})$ and can be corrected by replacing with the computed indices $(I_{\bar{X}}^i)$. Go to step 8.

**Step 5:** If all the $(\bar{D}_j)$, $j=2, 3, \ldots, \ell+1$ have equal number of nonzero residue digits corresponding to the moduli $m_{n-t+2}, m_{n-t+3}, \ldots, m_{n+1}$, for $0 < t < \ell$ and $(\bar{D}_1)$ has atmost $(\ell-t)$ nonzero residue digits corresponding to the moduli $m_2, m_3, \ldots, m_{\ell-t}, m_{\ell-t+1}$, then the residue digits $\bar{x}_{n-t+2}, \bar{x}_{n-t+3}, \ldots, \bar{x}_{n+1}$ and the first $(\ell-t)$ residue digits of the index $(\overline{I_X^1})$ are erroneous.

**Step 6:** These errors are corrected as

$$x_{n-t+k} = \left| \bar{x}_{n-t+k} - \left| m_j . (\bar{D}_j) \right|_{m_{n-t+k}} \right|_{m_{n-t+k}},$$

$k=2, 3, \ldots, t+1$ and for same $j, 2 \leq j \leq \ell+1$. compute a new index of the modulus $m_1$ and replace the erroneous index $(\overline{I_X^1})$ with it. Go to step 8.

**Step 7**: The errors lie among the residue digits $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}$. These erroneous residue digits are corrected as

$$x_q = \left| \bar{x}_q - |m_i . (\bar{D}_i)|_{m_q} \right|_{m_q}, \text{ where } m_q$$

are those moduli whose corresponding residue digits in $(\bar{D}_i)$ are nonzero.

**Step 8:** Use $x_1, x_2, \ldots, x_n$ residue digits to compute $x'_{n+1}$ corresponding to R-modulus $m_{n+1}$ by using the BEO. If $x'_{n+1} \neq x_{n+1}$, then an overflow is detected; otherwise the number X is in the legitimate range. Stop.

(B) Single Residue Digit-Error with Overflow:

For correcting single residue error, we represent $X \varepsilon [0, 2M-1)$ with first two magnitude indices $(I^1_X)$ and $(I^2_X)$ as

$$X \leftrightarrow (x_1, x_2, \ldots, x_{n+1}, (I^1_X), (I^2_X)).$$

Let $\bar{X} \leftrightarrow (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}, (\overline{I^1_X}), (\overline{I^2_X}))$ be the redundant representation of an erroneous number $\bar{X} \varepsilon [0, M.m_{n+1})$ having only single residue digit error. Compute the indices $(I^1_{\bar{X}})$ and $(I^2_{\bar{X}})$ with the help of the residue digits $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}$ and define

$$(\bar{D}_i) = ((I^i_{\bar{X}}) - (\overline{I^i_X})), \ i = 1, 2.$$

**Theorem 4**

Assuming that only one residue digit error occurs the following assertions hold true provided $\text{Min.}(m_f . m_g) \geq m_2$.

$$1 \leq f, g \leq n+1$$

**Assertion 1**

If only one $(\bar{D}_i) \neq (0)$, then the residue, digits $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+1}$ are correct and the error lies in the index $(\overline{I^i_X})$. The error is corrected by replacing $(\overline{I^i_X})$ with the computed index $(I^i_{\bar{X}})$.

**Assertion 2**

If atleast one of $(\bar{D}_i)$ has only one nonzero residue digit corresponding to a modulus, say $m_k$, $1 \leq k \leq n+1$, $k \neq i$, and rest of $(\bar{D}_i)$ have atleast $(n-1)$ nonzero residue digits, then the indices are correct and $\bar{x}_k$ is an erroneous residue digit. It is corrected as

$$x_k = \left| \bar{x}_k - |m_i . (\bar{D}_i)|_{m_k} \right|_{m_k}.$$

**Assertion 3**

If all the $(\bar{D}_i) = (0)$, then there is no error.

**Proof**

These assertions can also be proved in the same manner.

By using these assertions, we can correct a single residue error.

**Remark**

This method can correct any $\ell$ errors such that $\frac{n-1}{2} < \ell \leq n$, provided if we add $(2\ell+1-n)$ R-moduli to the given set of the moduli and consider only first $2\ell$ indices.

## Comparison

We have proposed a new approach for correcting multiple errors alongwith the detection of an additive overflow in residue code. This approach results in faster decoding procedures and has following advantages over those given by Mandelbaum[14], Barsi & Maestrini[5,6] and Arora & Saroj[2].

1. The procedures for single, burst and multiple residue error correction based upon the proposed approach operate on the residue representation of the numbers and make exclusive use of the modular operations in contrast to the procedures suggested by Barsi & Maestrini[5], Mandelbaum[14], which require a separate positional processor for implementation.

2. Barsi & Maestrini[6] has suggested algorithm for single error detection & correction with magnitude index whereas the proposed approach is general for $\ell (\leq n)$ number of errors. The definition of magnitude index in this paper is different from the one introduced by Barsi & Maestrini[6]. This definition leads to faster decoding algorithms as compared to the one suggested by Barsi & Maestrini at the cost of redundancy. But in an era of VLSI technology, it is always desirable to have faster decoding algorithms.

3. If $\frac{n-1}{2} < \ell \leq n$, then the proposed approach needs $(2\ell-n+1)$ R-moduli whereas the method suggested by Arora & Saroj[2] requires atleast $(2\ell+1)$ R-moduli.

The proposed procedures are faster as compared to those suggested by Arora & Saroj[2].

Table I shows the number of modular operations required by the proposed procedures (P) and the procedures (Q) suggested earlier by author in [2]. Thus we see from the table that the proposed approach results in the faster decoding procedures.

## Conclusion

In this paper, a new approach for correcting multiple errors along with detection of an additive overflow in residue code is presented. This approach works with the code whose redundancy is in the form of magnitude indices. Further,

Table I

| Type of Errors | Procedure | Number of Modular Operations Required |
|---|---|---|
| Multiple | Procedure (Q) | $\binom{n+\ell+1}{n+1}.(2n+3r-2)+(2n+3r-1)$ |
| | Procedure (P) | $2n + 8 + 1$ |
| Burst | Procedure (Q) | $\left\lfloor \frac{n+\ell+1}{+1} \right\rfloor^{*}.\ (2n+2r+2\ell-1) + (4n+6r-3)$ |
| | Procedure (P) | $2n + 5\ell + 4$ |
| Single | Procedure (Q) | $\lceil (n+1)/2 \rceil^{**}.(2n+2r+1)+(4n+6r-3)$ |
| | Procedure (P) | $2n+9$ |

procedures for special cases, viz., single burst residue error and single residue error are also suggested. The advantages of this approach have been stated over those of Arora & Saroj[2], Barsi & Maestrini[5,6] and Mandelbaum[14].

## References


[1] Aiken, H. and Semon, W.,"Advanced Digital Computer Logic, Computing Lab., Harvard Univ.,Cambridge, Mass., Report No. WADS, 1959.

[2] Arora, R.K. and Sharma, Saroj,"Correction of Multiple Errors and Detection of Additive Overflow in Residue Code, Information and Control, vol.39, No.1, pp. 46-54, 1978.

[3] Banerji, D.K., 'Residue Arithmetic in Computer Design," Ph.D. thesis, Dept. of Computer Science, Univ. of Waterloo, Waterloo,Ontario, Canada, 1971.

[4] Banerji, D.K. and Kaushik,Saroj, "On combinational Logic for Sign Detection in Residue Number Systems, Australian Computer Journal, vol.16, No.3, pp.90-95, 1984.

[5] Barsi,F. and Maestrini,P.,"Error Detection and Correction by Product Codes in Residue Number Systems," IEEE Trans. on Computers, vol.C-23, pp. 915 - 924, 1974.

[6] Barsi,F. and Maestrini, P.,"Arithmetic Codes in Residue Number Systems with Magnitude Index," IEEE Trans. on Computers, vol.C-27, pp. 1185-1188, 1978.

[7] Chien,R.T.,"Error Correction in High Speed Arithmetic", IEEE Trans. on Computers,vol.21,pp.433,1972.

[8] Diamond,R.M.,"Checking Codes for Digital Computers," Proc.IRE,vol.43, 1955.

[9] Garner,H.L.,"The Residue Number System," IRE Trans.on Electronic Computers,vol.EC-8,pp.140-147,1959.

[10] Kaushik,Saroj,"On the Arithmetic Operations and Error Correction in Residue Code;" Ph.D.thesis, Indian Institute of Technology,Delhi,India, 1980.

[11] Kaushik Saroj,"Sign Detection in Non-Redundant Residue Number System with Reduced Information",6th Symposium on Computer Arithmetic,Aarhus,Denmark, 1983.

[12] Kaushik Saroj,"Sign Detection in Residue Code",Computers and Elec Engg, (Accepted)-1984.

[13] Mandelbaum,D.,"Error Correction in Residue Arithmetic", IEEE Trans.on Computers,vol.C-21,pp.538-545.

[14] Mandelbaum,D.,"On a class of Arithmetic Codes and A Decoding Algorithm," IEEE Trans.on Information Theory,vol. IT-12,pp.85-88,1976.

[15] Rao,T.R.N.,"Error Checking logic for Arithmetic Type Operations of a Processor",IEEE Trans.on Computers,vol. C-19,pp.752-747, 1968.

[16] Sasaki,A.,"Addition and Subtraction in the Residue Number System,"IEEE Trans.Electronic Computers,vol.EC-16, pp.157-164,1967.

[17] Svoboda,A.,"The Numerical System of Residual Classes (SRC),Digital Information Processors,Walter Hoffman:John Wiley,1962.

[18] Szado,N.S. and Tanaka,R.I.,"Residue Arithmetic and its Applications to Computer Technology",New York:McGraw-Hill, 1967.


* $\lfloor I \rfloor$ denotes the floor of I i.e. the largest integer $\leq$ I.

** $\lceil I \rceil$ denotes the ceiling of I i.e. the smallest integer $\geq$ I.