FINITE PRECISION LEXICOGRAPHIC CONTINUED FRACTION NUMBER SYSTEMS

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Abstract

Lexicographic continued fraction binary (LCF) representation provides an order preserving bitstring representation of the non negative real numbers where every rational number has a finite length bitstring representation. We investigate the precision of k-bit LCF approximation. The maximum gap size over [0,1] for (k+1)-bit LCF representation is shown to be less than $2^{-.81k}$, comparable to binary coded decimal in worst case representation efficiency. The distribution of gap sizes for (k+1)-bit LCF representation over [0,1] is shown on a logarithmic scale to be bell shaped between $2^{-.81k}$ and $2^{-1.39k}$, becoming more peaked near the value corresponding to uniform spacing, 2^{-k}, with increasing k.

I. Introduction and Summary

For the purposes of computer arithmetic it is appropriate to view a number system as an abstract data type (ADT), where the specification of the set of arithmetic operations to be supported serve to characterize the ADT. The choice of a numeric representation and corresponding binary format then constitutes more than simply an encoding of numeric data. Numeric representation should be likened to the specification of a data structure to support the ADT. Questions of space efficiency of the representation and the time efficiency of procedures applied to the representation to realize the particular set of arithmetic operations characterizing the ADT are then criteria by which the suitability of the data representation must be judged.

The availability of VLSI and computer manufacturers' recent interest (e.g. [IEEE81], [KM84]) in providing firmware/hardware realizations of a variety of numeric representations (characterized by their ADT's) has brought renewed interest in the investigation of alternative numeric representations. In this paper we continue our investigation of a new binary numeric representation that ideally supports a number system with arithmetic operations synergistically allowing both exact rational arithmetic and approximate real arithmetic.

The lexicographic continued fraction binary (LCF) representation introduced in [MK83] provides an order-preserving one-to-one correspondence between the reals and all infinite bitstrings (excluding as in binary radix representation those bitstrings with a terminating infinite sequence of units). It was shown in [MK83] that LCF representation provides a space efficient finite bitstring encoding (with implied terminating infinite sequence of zeroes) for every rational number, and supports time efficient procedures for exact and/or approximate arithmetic (+, −, ×, ÷, <, =) as implicit from [KM83] and [Go72]. Our concern in this paper is to investigate the suitability of k-bit LCF representation in support of approximate real arithmetic, in particular, to study the uniformity of approximation error obtained by rounding to k-bit LCF numbers. For this purpose we first develop appropriate tools for characterizing and analysing the nature and size of gaps between representable k-bit LCF numbers. Given the variability of gap sizes between neighboring k-bit LCF representable numbers evident from enumerating the representable values for small k, we are then especially interested in determining the maximum gap size and distribution of gap sizes as a function of k for large k.

In Section II we review the definition and primary properties of LCF representation from [MK83]. We introduce the LCF tree as a convenient reference for displaying fractions in correspondence with their LCF representation, and for assessing the precision obtainable by k-bit LCF approximation.

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Our main result described in Section III derives from the formulation of an operation termed the "binary mediant" and provides for convenient development of the LCF tree and concomitant analysis of the nature and size of the gaps between neighboring k-bit LCF numbers. Specifically, if the irreducible fractions p/q and r/s are neighboring k-bit LCF numbers where say $p \le r$, then there must exist a unique largest integer j such that $2^j p \le r$ and $2^j q \le s$. The irreducible fraction u/v, termed the binary mediant of p/q and r/s, which is the (k+1)-bit LCF representable number falling between p/q and r/s is then given by

$$\frac{u}{v} = \begin{cases} \dfrac{2^j p + r}{2^j q + s} & \text{if} \quad |ps - qr| = 1, \\[2ex] \dfrac{(p+r)/2}{(q+s)/2} & \text{if} \quad |ps - qr| \ne 1. \end{cases} \tag{1}$$

Throughout this paper we use equality of fractions, a/b = c/d, in the stronger sense of ordered pair equality, i.e. to denote identical numerator values (a = c) and denominator values (b = d). When the numerator and/or denominator is itself an expression, the horizontal line format is used to clearly separate the fraction's numerator expression from the denominator expression as in the preceding formula for u/v.

In Section IV we employ the binary mediant and simulations to show that the gap size variability over the (k+1)-bit LCF numbers is reasonably narrow. For (k+1)-bit representation where the 2^k-1 representable values within [0,1] would effect a uniform spacing of size 2^{-k} if evenly spaced, we show that the maximum gap size is bounded by $2^{-.81k}$. By exhaustive enumeration through k = 24 and by simulation to k = 128, we show the distribution of gap sizes on the logarithmic scale to be bell shaped very narrowly peaked for large k near 2^{-k}, with vanishingly small tails truncated at points corresponding to the maximum gaps of size $2^{-.81k}$ and minimum gaps of size $2^{-1.39k}$. E.g. for k = 64 fully 99.9% of all gaps have size between $2^{-.89k}$ and $2^{-1.13k}$. Consider that binary coded decimal representation requires k = 4p bits to provide uniform spacing of size $1/10^p$ over [0,1]. Since $10^{-p} = 2^{-(\log_2 10)k/4} = 2^{-.83k}$, we note that the very infrequent largest gap size in k-bit LCF representation yields essentially no worse representation efficiency than that achieved throughout the unit interval for binary coded fixed-point decimal representation.

II. Lexicographic Binary Representation of Integers and Rationals

Utilizing the notation $[a_0/a_1/a_2/ \dots]$ for the continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} \qquad a_i \ge 0$$

where the partial quotients a_i are assumed to be integral, it is known from the theory of continued fractions [HW79] that any non-negative rational number, denoted by the irreducible fraction p/q, has a finite expansion which is unique in terminal index even form

$$\frac{p}{q} = [a_0/a_1 \dots /a_{2m}] \tag{2}$$

with the added requirements $a_0 \ge 0$, $a_i \ge 1$ for $1 \le i \le 2m$.

For the purpose of defining a continued fraction based binary representation of rationals we first provide representations of the positive integers (dealing separately with the case $a_0 = 0$) which are "self-delimiting", i.e. which implicitly contain end-markers when read left-to-right. The self-delimiting binary integer representation employed utilizes the standard binary representation preceded by a unary encoding of its length. Concatenation of such integer representations of the partial quotients of a continued fraction provides a binary representation where every rational number will have finite length. The LCF representation introduced in [MK83] further provides for the lexicographic order of the representations to correspond to numeric order of the rational values.

Formally, if the integer $p \ge 1$ has the (n+1)-bit base 2 radix representation $b_n b_{n-1} \dots b_1 b_0$ with $b_i \in \{0,1\}$, $b_n = 1$, then*

$$\ell(p) = 1^n \circ 0 \circ b_{n-1} b_{n-2} \dots b_1 b_0 \tag{3}$$

will be termed the lexicographic integer representation of p. This representation is order preserving in that the lexicographic ordering (leftmost bit first) of the bit strings corresponds to numeric ordering of the values. Table 1 contains the $\ell(\cdot)$ representations of the first 20 positive integers. For a discussion of various lexicographic order preserving binary encodings of the integers, see [Kn80] where similar representations are analysed.

*Here and in the following o denotes string-concatenation.

Integer	Standard Binary	Lexicographic	Integer	Standard Binary	Lexicographic
1	1_2	0	11	1011_2	1110011
2	10_2	100	12	1100_2	1110100
3	11_2	101	13	1101_2	1110101
4	100_2	11000	14	1110_2	1110110
5	101_2	11001	15	1111_2	1110111
6	110_2	11010	16	10000_2	111100000
7	111_2	11011	17	10001_2	111100001
8	1000_2	1110000	18	10010_2	111100010
9	1001_2	1110001	19	10011_2	111100011
10	1010_2	1110010	20	10100_2	111100100

Table 1. Right-adjusted standard binary representation and left-adjusted lexicographic bitstring representation of the integers 1, 2, ..., 20.

Note from the definition that the continued fraction expansion of $p/q = [a_0/a_1/a_2/ \dots /a_{2m}]$ is an increasing function of the even indexed partial quotients, and a decreasing function of the odd indexed quotients. Thus to obtain an order preserving representation of the rationals we simply represent the odd indexed quotients in complement form before concatenation. To be able to compare bit strings lexicographically from left to right it is assumed that any (finite length) representation is extended to the right with an arbitrary number of extra zeroes.

Formally, implicitly handling the case $a_0 = 0$ $(0 \le p/q < 1)$ by a leading zero bit and $a_0 \ge 1$ $(1 \le p/q)$ by a leading unit bit,

$$\mathrm{LCF}\left(\frac{p}{q}\right) = \begin{cases} 1 \circ \ell(a_0) \circ \overline{\ell(a_1)} \circ \ell(a_2) \circ \dots \circ \overline{\ell(a_{2m-1})} \circ \ell(a_{2m}) \circ \overline{\ell(\infty)} & \text{for } 1 \le \frac{p}{q}, \\ 0 \circ \overline{\ell(a_1)} \circ \ell(a_2) \circ \dots \circ \overline{\ell(a_{2m-1})} \circ \ell(a_{2m}) \circ \overline{\ell(\infty)} & \text{for } 0 \le \frac{p}{q} < 1, \end{cases} \quad (4)$$

where the finite bitstring form of LCF(p/q) is truncated at the last unit bit, except for LCF(0) whose finite form will be composed of a single 0.

<u>Example</u>

$$\frac{22}{7} = [3/6/1] \quad \{\text{terminal index even continued fraction form}\}$$
$$= [3/6/1/\infty] \quad \{\text{infinite extension}\}$$

$$\begin{aligned} \mathrm{LCF}\left(\frac{22}{7}\right) &= 1 \circ \ell(3) \circ \overline{\ell(6)} \circ \ell(1) \circ \overline{\ell(\infty)} \\ &= 1 \circ 101 \circ \overline{11010} \circ 0 \circ \overline{11\dots} \\ &= 1 \circ 101 \circ 00101 \circ 0 \circ 00\dots \\ &= 110100101 \quad \{\text{finite form}\} \end{aligned}$$ []

Observations concerning the LCF-representation given in [MK83] are summarized here:

— There is a one-to-one correspondence between all finite bitstrings terminating in a unit and all positive irreducible fractions.

— The LCF-representation extends naturally to the positive reals.

— The LCF-representation is order preserving over the non-negative reals, i.e. lexicographical ordering of LCF-strings corresponds to numerical ordering of the values.

— LCF(q/p) is the 2's complement of LCF(p/q) whenever $p > 0$, so the inverse of a k-bit LCF number is a k-bit LCF number.

— The bit pattern in LCF(p/q) may be interpreted as an encoding of the steps (transitions) in a finite automaton performing the Euclidean GCD-Algorithm (in binary) on p and q.

— LCF-representations may be used bit-by-bit left-to-right as input to on-line algorithms, for the direct computation of arithmetic expressions upon LCF-represented operands yielding LCF-represented results [KM83], [Go72].

— Signed LCF representation is given by:

$$\mathrm{SLCF}(p/q) = \begin{cases} 1 \circ \mathrm{LCF}(p/q) & \text{for } 0 \le p/q \\ 0 \circ \overline{\mathrm{LCF}(-p/q)} & \text{for } p/q < 0 \end{cases}$$

where $\overline{\mathrm{LCF}}$ denotes the 2's complement of the LCF bit string. Note that SLCF representation is then order preserving over the reals.

LCF representation is conveniently illustrated for all fractions hierarchically in terms of finite bitstring length by associating the positive irreducible fractions with the nodes of an infinite binary tree, termed the <u>LCF tree</u>, where the LCF bitstring denotes the path to the node containing the associated irreducible fraction. For LCF(p/q) given by the k-bit string $b_1 b_2 \dots b_{k-1} 1$, the fraction p/q is assigned to the node at depth k-1 reached by proceeding to the left child when $b_i = 0$ and to the right child when $b_i = 1$ for $i = 1,2,\dots,k-1$.

The LCF tree illustrated to depth 4 in Figure 1 provides a convenient reference for interpreting the accuracy of finite precision LCF representation both with regard to the increasing precision obtained by a k-bit LCF approximation of a specific number as k increases, and with regard to the ordered enumeration of all k-bit LCF representable values for any fixed k. Specifically,

III. The Binary Mediant and LCF Representation

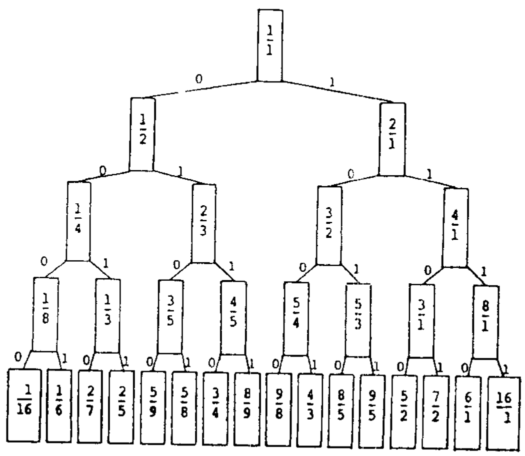

Figure 1. The LCF tree through depth four.

- Fractions in the nodes along a path determined by an LCF represented value provide successively tighter bounds on the value. E.g. the first five bits of the LCF number 01011... indicate a value in the interval [5/8, 2/3].
- Inorder enumeration of the fractions in the nodes of the LCF tree truncated at depth k-1 provides the ordered list of all positive rationals representable by k-bit LCF bitstrings. The seven positive 3-bit LCF numbers are then 1/4, 1/2, 2/3, 1/1, 3/2, 2/1, 4/1.

From the LCF tree it is further observed to the depth illustrated that:

- Each gap between representable positive k-bit LCF numbers is at most 2/3 the size of the gap between the (k-1)-bit LCF numbers in which it fell, at most 2/5 the size of the (k-2)-bit gap in which it fell, and at most 2/9 the size of the (k-3)-bit gap in which it fell.
- The irreducible fraction u/v associated with any node in the LCF tree can be conveniently determined as follows. If p/q, r/s are the irreducible fractions in the nearest left and right ancestor nodes where say $p \le r$, and if j is the largest integer such that $2^j p \le r$ and $2^j q \le s$, then $\frac{u}{v} = \frac{2^j p+r}{2^j q+s}$ for $|ps - qr| = 1$, and $\frac{u}{v} = \frac{(p+r)/2}{(q+s)/2}$ for $|ps - qr| \ne 1$.
- If p/q is the fraction in the parent node of the node containing r/s in the LCF tree, then $.75\, p \le r \le 2p$ and $.75\, q \le s \le 2q$.

These latter observations hold for the whole LCF tree. Their formalization and application to the investigation of the accuracy of finite precision LCF representation are developed in the next sections.

The number theoretic concepts of the "mediant" of two fractions and the "adjacency relation" on fractions provided a foundation for our previous investigation and development of fixed-slash and floating-slash number systems [MK80, KM83, MK85]. The purpose of this section is to introduce and develop the analogous concept of "binary mediant" to provide a foundation for investigation of LCF representation.

Recall that fixed-slash number systems consist of Farey-sets

$$F_{2^n} = \left\{ \frac{p}{q} \,\middle|\, 0 \le p,\ q < 2^n,\ \gcd(p,q) = 1 \right\}$$

whose members may be described by recursively introducing the mediant (p+r)/(q+s) of two successive rationals p/q < r/s already in the set, starting with 0/1 and 1/0. Restricting analysis to the interval $0 \le p/q \le 1$, it can be shown [MK80] that the size of gaps between successive members of F_{2^n} vary between 2^{-n} and 2^{-2n}, the larger gaps bounded on one side by rationals whose numerator and denominator are relatively small integers (intuitively termed simple fractions). It is easily seen that if say p/q is a very simple fraction, whereas r/s is not (e.g. p<<r, q<<s), then their mediant will be of numerical value very close to r/s but rather distant from p/q. If, however, p and q were both multiplied by some common factor c, chosen such that cp and cq were both of the same order of magnitude as r and s respectively, then the rational value (cp+r)/(cq+s) would split the interval between p/q and r/s into two intervals of almost equal width.

This observation leads us to the introduction of an alternative form of mediant more nearly bisecting each gap which is conveniently related to binary representation. For any two non-negative fractions p/q and r/s we first define the partial ordering [MK80]:

$$\frac{p}{q} \lessdot \frac{r}{s} \quad \text{iff} \quad \frac{p}{q} \ne \frac{r}{s} \text{ and } p \le r \text{ and } q \le s, \qquad (5)$$

where the symbol $\lessdot$ denotes simpler than. For the irreducible fractions $p/q \lessdot r/s$, let j be the largest integer such that $2^j p/2^j q \lessdot r/s$. Then the binary mediant of p/q and r/s is the irreducible fraction u/v with value given by $(2^j p+r)/(2^j q+s)$.

It is important to note that $\gcd(2^j p+r,\ 2^j q+s)$ is not necessarily unity, but the binary mediant is the reduced fraction u/v with the same value so then $\gcd(u,v) = 1$ by definition.

The relationship between the binary mediant and LCF representation is developed in the following lemmas which for brevity are given without proofs.

Lemma 1: If $p/q = [a_0/a_1/\ldots/a_n]$ and $r/s = [a_0/a_1/\ldots/a_n+2^i]$ where $i>0$ and $2^i \leq a_n$, then the binary mediant of p/q and r/s is

$$\frac{u}{v} = \frac{(p+r)/2}{(q+s)/2} = [a_0/a_1/\ldots/a_n+2^{i-1}].$$

Lemma 2: If $p/q = [a_0/a_1/\ldots/a_n]$, $r/s = [a_0/a_1/\ldots/a_n/2^i]$, $i\geq 0$, then the binary mediant of p/q and r/s is

$$\frac{u}{v} = \frac{2^i p+r}{2^i q+s} = [a_0/a_1/\ldots/a_n/2^{i+1}].$$

Lemma 3: If $p/q < r/s$ have LCF-representations $LCF(p/q) = \sigma \circ 0 \circ 1^j$ and $LCF(r/s) = \sigma \circ 1 \circ 0^j$ for $j\geq 0$, and u/v is the binary mediant of p/q and r/s, then

$$LCF(\frac{u}{v}) = \sigma \circ 0 \circ 1^{j+1}.$$

In Lemmas 1-3 we refer to a continued fraction expansion of a fraction without being specific about any canonical form. From continued fraction theory [HW79] any positive fraction will have exactly two continued fraction expansions under the assumption $a_i \geq 1$ for all $i \geq 1$, which are related as follows:

$$\frac{p}{q} = \begin{cases} [a_0/a_1/\ldots/a_n] & (6) \\ & \text{for } a_n \geq 2. \\ [a_0/a_1/\ldots/a_n-1/1] & (7) \end{cases}$$

Formula (6) is generally chosen as the canonical form, and we have earlier chosen between (6) and (7) to select that form whose terminal index is even simply to guarantee that the LCF representation has a terminal string of 0's. Note that utilizing the corresponding terminal index odd continued fraction and encoding into LCF form would yield a bit string differing only in that $LCF(p/q) = \sigma 100\ldots$ would become $\sigma 011\ldots$ for the appropriate substring σ. As in ordinary radix representation we take these bit strings as equivalent preferring to write $LCF(p/q) = \sigma 1$, assuming the finite sequence is extended to the right by an infinite string of zeros.

Lemma 4: The labeling of the nodes of the LCF-tree can be obtained by succesively generating the fraction occurring as label, as the binary mediant of its nearest left and right ancestors. The labeling of the edges (0 for left branch, and 1 for right branch) read as a path to a node provides the LCF representation of the fraction at that node, when completed by a unit-bit.

IV. The Distribution of Gap Sizes in the k-bit LCF System

To investigate the distribution of gap sizes between the k-bit LCF rationals, consider the following. The LCF rationals in the interval from one to infinity are the reciprocals of the LCF rationals in the interval zero to one. Hence we shall restrict our analysis to the unit interval. Also note that in the closed unit interval there are exactly 2^k+1 different rationals in the (k+1)-bit LCF system, the leading bit being used to distinguish that the number falls in the unit interval, rather than in the interval from one to infinity. Equivalent uniform spacing as in fixed-point binary would yield a uniform gap size of 2^{-k}, but would not provide for exact representation of all the relatively simple rationals.

For purposes of comparing the space efficiency of LCF representation we are concerned with the size k of the finite length k-bit LCF representation of p/q in comparison with the size of p and q. More critically for purposes of uniformity of approximation we are concerned with the extent of variability of the gap sizes in k-bit LCF representation about the ideal value 2^{-k} with increasing k.

Notice that the growth of the values of p and q, as a function of the position of the last non-zero bit in LCF(p/q), is dependent on the encoding of the partial quotients of the continued fraction expansion of p/q, i.e. on the encoding $\ell(\cdot)$ specified in Section II. It is known from the measure theory of continued fractions (e.g. [Kh63] or [Kn81]) that for p/q chosen uniformly on [0,1], the probability that any particular partial quotient a_i of p/q takes a specific value j depends only on j (for sufficiently large i) and is given by

$$P(a_i = j) = \log_2 \frac{(j+1)^2}{j(j+2)},$$

hence $P(a_i = 1) = 41.5\%$, $P(a_i = 2) = 17.0\%$, $P(a_i = 3) = 9.3\%$, $P(a_i = 4) = 5.9\%$, etc.

From information theoretic considerations an object occurring with probability p is best encoded by using $-\log_2 p$ bits. Ideally then we should use 1.27 bits for encoding 1, 2.56 bits for encoding 2, 3.42 bits for 3, and 4.09 bits for 4. Since $\ell(1) = 0$, $\ell(2) = 100$, $\ell(3) = 101$ and $\ell(4) = 11000$, we observe that 1 has a rather short representation when compared to the frequency with which it occurs, whereas 2 and 4 have quite long representations. Hence we may expect that the numerator and denominator of $p/q = [0/1/1/\ldots/1]$, where $LCF(p/q) = 0101\ldots01$, will exhibit a fast growth rate as a function of k, whereas $r/s = [0/2/4/\ldots/2/4]$, where $LCF(r/s) = 0011110000 1111\ldots00001111$, will have a slow growth of r and s.

p and q will grow asymptotically as $2^{1.3885k}$, for LCF(p/q) = 0101...01, and similarly will be of the order $2^{0.8268k}$ for LCF(r/s) = 0011110000l111... 00001111. Based on these observations we may also expect asymptotically a minimal gap size between k-bit LCF-numbers to be at least as small as $2^{-1.3885k}$ and a maximal gap size to be at least as large as $2^{-0.8268k}$, where 2^{-k} would have been obtained if the gaps were all of the same width. Utilizing the notion of binary mediant from the preceding section we are able to derive an overall upper bound on the maximum gap size. For brevity we include here only the main idea of the proof.

<u>Theorem 5:</u> For any $k \geq 9$, the largest gap size over [0,1] between consecutive (k+1)-bit LCF numbers is less than $2^{-0.81k}$.

<u>Sketch of Proof</u>: Let the "worst case" effective precision α be defined by having $2^{-\alpha k}$ equal the largest gap size over [0,1] in (k+1)-bit LCF representation. By direct computation we obtain the following values of α for k = 1,2,...,20.

k	α	k	α
1	1.000	11	0.812
2	0.792	12	0.816
3	0.774	13	0.819
4	0.792	14	0.820
5	0.817	15	0.816
6	0.812	16	0.819
7	0.804	17	0.821
8	0.810	18	0.822
9	0.815	19	0.818
10	0.818	20	0.821

In general the LCF representation of any number $0 < x < 1$ may be given in the form

$$x = 0^{n_1} 1^{n_2} 0^{n_3} 1^{n_4} \ldots$$

composed of blocks of $n_i \geq 1$ identical bits (0's or 1's) between reversals. Let r/s be the last node value on the path in the LCF tree corresponding to x before the i^{th} block and let p/q be the preceding (parent) node value. Then x is known to be in a gap of size $|p/q - r/s|$ before interpreting the i^{th} block of bits. For the case where $p/q \leqslant r/s$ it follows that $r/s \leqslant 2p/2q$. Then the value in the node corresponding to reading j of the next n_i bits is the irreducible fraction of value given by the not necessarily reduced fraction $((2^j-1)p+r)/((2^j-1)q+s)$ for $j = 1,2,\ldots,n_i$. After proceeding through the i^{th} block x will then be known to fall in a gap of size

$$\left| \frac{(2^{n_i}-1)p+r}{(2^{n_i}-1)q+s} - \frac{(2^{n_i-1}-1)p+r}{(2^{n_i-1}-1)q+s} \right|$$

as compared to the previous gap size $|p/q-r/s|$. Taking the maximum ratio of new to old gap size for each value of $n=n_i$ over all $1 \leq s/q \leq 2$ yields 1/2 for n=1, $1/(2+\sqrt{3})$ for n=2, and $2^n/[(2^n+1)(2^{n-1}+1)]$ for $n\geq 3$. The effectiveness per bit of a string of n similar bits between reversals is then no worse than the negative base two logarithm of this ratio divided by n, giving an average effectiveness per bit of 0.8143 for n=4 as the worst case.

The details of handling the case where $r/s \leqslant p/q$ and of treating a terminal segment (no implied reversal) can be handled to verify the theorem. □

It should be noted that the argument of the preceding proof outline yields the LCF bit pattern 000011110000l111...00001111 as asymptotically achieving the minimum average reduction in gap size bounds per bit, consistent with the preceding heuristic argument that

LCF([0/2/4/.../2/4]) = 0011110000l111...00001111

would bound an interval of comparatively large gap size.

To further analyze the distribution of gap sizes a program was written to perform an exhaustive analysis of all gaps in the (k+1)-bit LCF systems up through k = 25. In each case it was found that the minimum size gap fell between two consecutive rationals of the form f(n-2)/f(n-1) and f(n-1)/f(n) where f(n) denotes the n^{th} Fibonacci number, in correspondence with the preceding observation about LCF([0/1/1/.../1]). It was also found that the maximum size gap occurred next to a rational whose LCF representation contained replications of the bitpattern 00001111, consistent with the preceding discussions.

The main purpose of the program was however to obtain a bar-graph of the distribution of gap sizes, as shown in Figure 2. Here the frequencies (in percent) of the occurrence of a particular gap size is plotted, as a function of the gap size measured relative to the uniform gap size of 2^{-k} for k = 6, 12, 18 and 24, in a logarithmic scale and using a bin size of 0.01. The initially erratic distribution for k=6 settles to a smooth bell shape for k=24, and the variation from the minimum gap size to the maximum gap size indicated by the truncated tails of the bell shape is essentially the same for k = 12, 18 and 24.

To illustrate the narrowing of the "bell-shape," simulations were run for higher values of k. In these simulations a random bitpattern was generated, and the equivalent rational number as well as a neighbor was determined. For each k chosen, 2^{21} gap sizes were determined, and their distribution plotted in a bar-graph as before. Simulated bar-graphs for k = 32,64,128 are shown in Figure 3, overlayed over the family of exhaustively computed distributions for k = 15, 16,...,25, confirming that the distribution gets narrower with a higher peak for increasing values of k.

Table 2 gives "effective precision" bounds a,b such that, respectively, 95%, 99.9%, and 100% of the gap sizes fall in the interval $[2^{-bk}, 2^{-ak}]$ with symmetric cutoffs, summarizing the narrowing of the peak with increasing k from the data of the preceding figures.

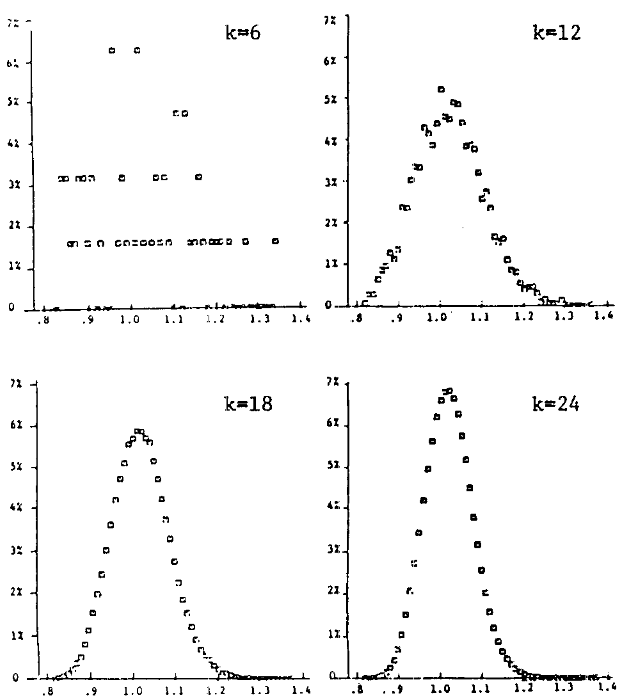

Figure 2: The percent of gaps in (k+1)-bit LCF representation over [0,1] that fall within the range $[2^{-ak}, 2^{-(a+.01)k}]$ for a = .80,.81,...,1.39 and k = 6,12,18,24, where values plotted for k = 12, 18 and 24 correspond exclusively to non zero percentages.

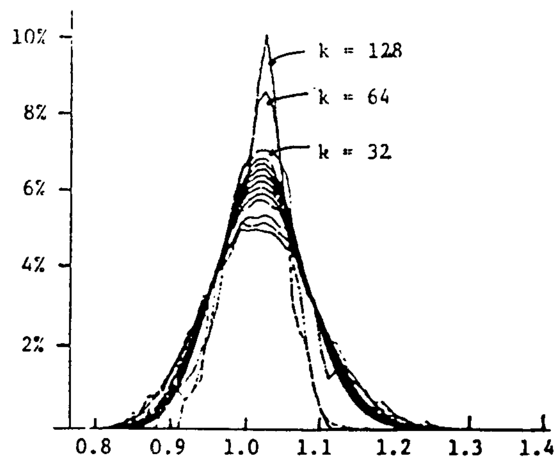

Figure 3: Overlay of the smoothed bar graph data for k = 15, 16, ..., 25 computed exhaustively, and the curves for k = 32, 64, and 128 computed by simulation using a sample of 2^{21} randomly chosen gaps.

k	95%	99.9%	100%
16	0.89 - 1.17	0.83 - 1.27	0.81 - 1.38
24	0.92 - 1.15	0.86 - 1.23	0.82 - 1.38
32	0.93 - 1.14	0.88 - 1.21	0.82 - 1.38
64	0.94 - 1.10	0.89 - 1.13	0.82 - 1.38
128	0.95 - 1.09	0.90 - 1.12	0.82 - 1.38

Table 2. Bounds a-b such that the indicated percentage of all gap sizes in (k+1)-bit LCF representation over [0,1] fall in the interval $[2^{-bk}, 2^{-ak}]$.

Overall our study shows that the gaps between LCF-representable values are subject to a variation in size corresponding to a worst case 18% precision loss (or equivalently an 18% storage capacity loss) and a best case 38% precision gain in comparison with an equivalent fixed-point binary system with uniform gap sizes over the unit interval. This appears to be a reasonably small price to pay in return for achieving exact representation of all simple rationals in a convenient format supported by the number theoretic foundations of continued fractions and "best rational approximation". It should be noted that the locations of the extreme larger gaps and smaller gaps within [0,1] correspond to continued fractions with particular sequences of partial quotients related to their chosen $\ell(\cdot)$ representation. Thus the gap size extremes may be regarded as base dependent anomalies of the representation and of no apparent intrinsic natural value other than to bound the precision variation of the representation.

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